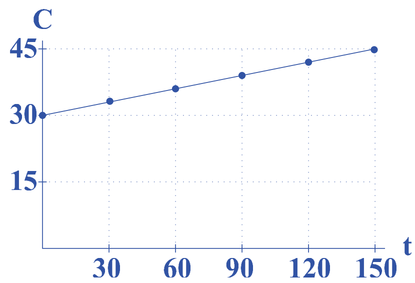
Sections 1.3 & 1.4 – Linear Functions and Their Formulas

Preliminary Example. The cost, C, of your monthly phone bill consists of a \$30 basic charge, plus \$0.10 for each minute of long distance calls.

(a) Complete the table below, and sketch a graph.

t (minutes)	0	30	60	90	120	150
C (\$)	30	33	36	39	42	45



(b) Compute the average rate of change of C over any time interval.

$$\frac{\Delta C}{\Delta t} = \frac{33-30}{30-0} = \frac{3 \text{ dollars}}{30 \text{ minutes}} = 0.10 \text{ dollars per minute.}$$

Note that this rate of change is the charge per minute of calling.

(c) Find a formula for C in terms of t.

$$\begin{aligned} C &= \$30 + \left(\frac{\$.10}{\text{minute}}\right)(\text{number of minutes talked}) \\ &= 30 + 0.10t, \end{aligned}$$

so our answer is $C = 30 + 0.10t$.

(d) If your bill is \$135, how long did you talk long distance?

We have

$$\begin{aligned} C = 135 \quad &\Longrightarrow \quad 135 = 30 + 0.10t \\ &\Longrightarrow \quad 105 = 0.10t \\ &\Longrightarrow \quad 1050 = t, \end{aligned}$$

which means that you talked 1050 minutes, or 17 hours and 30 minutes.

Notes on Linear Functions:

1. If $y = f(x)$ is a linear function, then $y = mx + b$, where

$$\begin{aligned} m &= \texttt{slope} \quad \leftarrow \quad \texttt{(rate of change of } y \texttt{ with respect to } x\texttt{)} \\ b &= y\texttt{-intercept} \quad \leftarrow \quad \texttt{(value of } y \texttt{ when } x = 0.\texttt{)} \end{aligned}$$

2. If $y = f(x)$ is linear, then equally spaced input values produce equally spaced output values.

Different forms for equations of lines:

1. $y = mx + b$ (m = slope, b = vertical intercept)
2. $y - y_0 = m(x - x_0)$ (m = slope, (x_0, y_0) is a point on the line)
3. $Ax + By + C = 0$ (A, B, and C are constant)

Example 2. Find the slope and the y-intercept for each of the following linear functions.

(a) $3x + 5y = 20$

Rewriting, we have

$$\begin{aligned} 3x + 5y = 20 \quad &\Longrightarrow \quad 5y = -3x + 20 \\ &\Longrightarrow \quad y = \frac{-3x + 20}{5} \\ &\Longrightarrow \quad y = -\frac{3}{5}x + 4, \end{aligned}$$

so the slope is $-3/5$ and the y-intercept is 4.

(b) $\dfrac{x - y}{5} = 2$

Rewriting, we have

$$\begin{aligned} \frac{x - y}{5} = 2 \quad &\Longrightarrow \quad 5 \cdot \frac{x - y}{5} = 5 \cdot 2 \\ &\Longrightarrow \quad x - y = 10 \\ &\Longrightarrow \quad y = x - 10, \end{aligned}$$

so the slope is 1 and the y-intercept is -10.

Examples and Exercises

1. Let $C = 20 - 0.35t$, where C is the cost of a case of apples (in dollars) t days after they were picked.

   (a) Complete the table below:

t (days)	0	5	10	15
C (dollars)	20	18.25	16.50	14.75

   (b) What was the initial cost of the case of apples?

   $$t = 0 \quad \Longrightarrow \quad C = 20 - 0.35(0) = 20,$$

   so the initial cost is \$20.

   (c) Find the average rate of change of C with respect to t. Explain in practical terms (i.e., in terms of cost and apples) what this average rate of change means.

   Since the cost function is linear, the rate of change of C with respect to t is constant, so we can choose any interval to do this calculation. Therefore, the average rate of change is

   $$\frac{\Delta C}{\Delta t} = \frac{18.25 - 20}{5 - 0} = -0.35 \text{ cents per day.}$$

   This means that for each day that goes by after the picking day, the price of the apples decreases by 35 cents.

2. In parts (a) and (b) below, two different linear functions are described. Find a formula for each linear function, and write it in slope intercept form.

   (a) The line passing through the points $(1, 2)$ and $(-1, 5)$.

   (b)

C	10	15	20	25
F	50	59	68	77

   (a) First we calculate the slope: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{5-2}{-1-1} = -\dfrac{3}{2}$

   Therefore, we know that the equation of the line looks like $y = -\frac{3}{2}x + b$. To obtain a final answer, we need to calculate b. Since $(1, 2)$ is a point on the line, we can substitute $x = 1$ and $y = 2$ into our formula to obtain

   $$y = -\frac{3}{2}x + b \qquad \Longrightarrow \qquad 2 = -\frac{3}{2} \cdot 1 + b \quad \Longrightarrow \quad 2 + \frac{3}{2} = b,$$

   which means that $b = \frac{4}{2} + \frac{3}{2} = \frac{7}{2}$. Our final answer is therefore $y = -\frac{3}{2}x + \frac{7}{2}$.

   (b) Letting F be the output variable and C be the input variable, we first calculate the slope: $m = \dfrac{\Delta F}{\Delta C} = \dfrac{59 - 50}{15 - 10} = \dfrac{9}{5}$

   Therefore, we know that the equation of the line looks like $F = \frac{9}{5}C + b$. To obtain a final answer, we need to calculate b. Since $(10, 50)$ is a point on the line, we can substitute $C = 10$ and $F = 15$ into our formula to obtain

   $$F = \frac{9}{5}C + b \qquad \Longrightarrow \qquad 50 = \frac{9}{5} \cdot 10 + b \quad \Longrightarrow \quad 50 - 18 = b,$$

   which means that $b = 50 - 18 = 32$. Our final answer is therefore $F = \frac{9}{5}C + 32$.

3. According to one economic model, the demand for gasoline is a linear function of price. If the price of gasoline is $p = \$3.10$ per gallon, the quantity demanded in a fixed period of time is $q = 65$ gallons. If the price is \$3.50 per gallon, the quantity of gasoline demanded is 45 gallons for that period.

   (a) Find a formula for q (demand) in terms of p (price).

   First, we calculate the slope:

   $$m = \frac{\Delta q}{\Delta p} = \frac{45 - 65}{3.5 - 3.1} = \frac{-20 \text{ gallons}}{0.4 \text{ dollars}} = -50 \text{ gallons per dollar}$$

   Therefore, we have $q = -50p + b$, so we need to find b. Since $q = 65$ when $p = 3.1$, we have

   $$q = -50p + b \quad \Longrightarrow \quad 65 = -50(3.1) + b \quad \Longrightarrow \quad 155 + 65 = b,$$

   so $b = 220$. Our final answer is therefore $q = -50p + 220$.

   (b) Explain the economic significance of the slope in the above formula. In other words, give a practical interpretation of the slope.

   The slope of -50 gallons per dollar indicates that, for every dollar increase in the price, the demand for gasoline goes down by 50 gallons.

   (c) According to this model, at what price is the gas so expensive that there is no demand?

   We have

   $$\begin{aligned} q = 0 \quad \Longrightarrow \quad 0 = -50p + 220 \quad & \Longrightarrow \quad 50p = 220 \\ & \Longrightarrow \quad p = 4.4. \end{aligned}$$

   Thus, at a price of \$4.40 per gallon, there is no demand.

   (d) Explain the economic significance of the vertical intercept of your formula from part (a).

   The vertical intercept occurs at the point $(0, 220)$. This indicates that, at a selling price of \$0 per gallon, the demand for gasoline would be 220 gallons. In other words, if the gas station gave the gas away for free, 220 gallons would be the demand.

4. Look back at your answer to problem 2(b). You might recognize this answer as the formula for converting Celsius temperatures to Fahrenheit temperatures. Use your formula to answer the following questions.

   (a) Find C as a function of F.

   Since $F = \frac{9}{5}C + 32 \quad \Longrightarrow \quad F - 32 = \frac{9}{5}C \quad \Longrightarrow \quad C = \frac{5}{9}(F - 32)$, our final answer is $C = \frac{5}{9}(F - 32)$.

   (b) What Celsius temperature corresponds to 90°F?

   Using part (a), we have $F = 90 \quad \Longrightarrow \quad C = \frac{5}{9}(90 - 32) = \frac{5}{9} \cdot 58 = 32.2^\circ$, so 32.2°C is our answer.

   (c) Is there a number at which the two temperature scales agree?

   The temperature scales will agree if and only if $F = C$, so we have

   $$\begin{aligned} F = C \quad \Longrightarrow \quad F = \frac{5}{9}(F - 32) \quad \Longrightarrow \quad 9F = 5F - 160 \quad & \Longrightarrow \quad 4F = -160 \\ & \Longrightarrow \quad F = -40, \end{aligned}$$

   so the scales agree at -40°. In other words, -40°F and -40°C are equal temperatures.