Section 1.2 – Rates of Change

Preliminary Example. The table to the right shows the temperature, T, in Tucson, Arizona t hours after midnight.

t (hours after midnight)	0	3	4
T (temp. in °F)	85	76	70

Question. When does the temperature decrease the fastest: between midnight and 3 a.m. or between 3 a.m. and 4 a.m.?

Even though the temperature drops more between midnight and 3 a.m. than it does between 3 a.m. and 4 a.m., the temperature decreases fastest between 3 a.m. and 4 a.m., as the calculations below demonstrate:

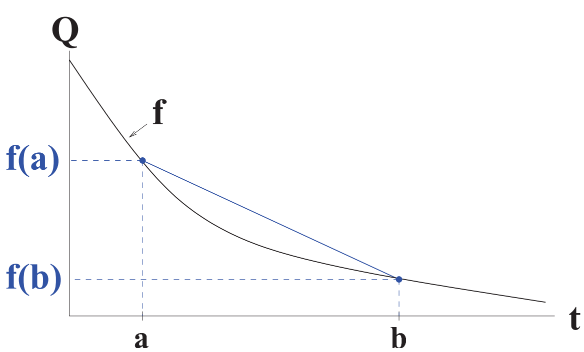
Interval	ΔT	Δt	$\frac{\Delta T}{\Delta t}$
$0 \leq t \leq 3$	-9°F	3 hours	$\frac{-9}{3} = -3$°F per hour
$3 \leq t \leq 4$	-6°F	1 hour	$\frac{-6}{1} = -6$°F per hour

Thus, the temperature decreases at an average rate of only 3°F per hour between midnight and 3 a.m., but it decreases at an average rate of 6°F per hour between 3 and 4 a.m.

Graphical Interpretation of Rate of Change

Definition. The **average rate of change**, or just **rate of change** of Q with respect to t is given by

$$\frac{\text{Change in } Q}{\text{Change in } t} = \frac{\Delta Q}{\Delta t}.$$



Alternate Formula for Rate of Change: The average rate of change of a function $Q = f(t)$ on the interval $a \leq t \leq b$ is given by the following formula:

$$\frac{f(b) - f(a)}{b - a}$$

Note: Graphically, the average rate of change given by the above formula is just the slope of the line segment in the above diagram.

Examples and Exercises

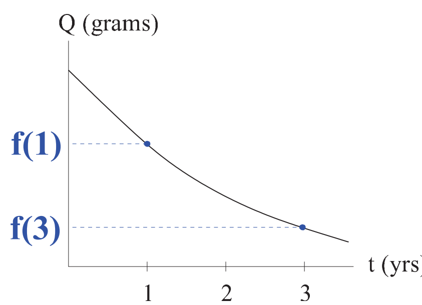
1. Let $f(x) = 4 - x^2$. Find the average rate of change of $f(x)$ on each of the following intervals.
   (a) $0 \le x \le 2$ (b) $2 \le x \le 4$ (c) $b \le x \le 2b$

   (a) We have
   $$\frac{f(2)-f(0)}{2-0} = \frac{(4-2^2)-(4-0^2)}{2} = \frac{0-4}{2} = -2,$$
   so the average rate of change of f is -2 over the interval $0 \le x \le 2$.

   (b) Similarly, we have
   $$\frac{f(4)-f(2)}{4-2} = \frac{(4-4^2)-(4-2^2)}{2} = \frac{-12-0}{2} = -6.$$

   (c) Similarly, we have
   $$\frac{f(2b)-f(b)}{2b-b} = \frac{(4-(2b)^2)-(4-b^2)}{b} = \frac{4-4b^2-4+b^2}{b} = \frac{-3b^2}{b} = -3b.$$

2. To the right, you are given a graph of the amount, Q, of a radioactive substance remaining after t years. Only the t-axis has been labeled. Use the graph to give a **practical interpretation** of each of the three quantities that follow. A practical interpretation is an explanation of meaning using everyday language.



   a. $f(1)$

   $f(1)$ represents the amount of the substance present, in grams, after 1 year.

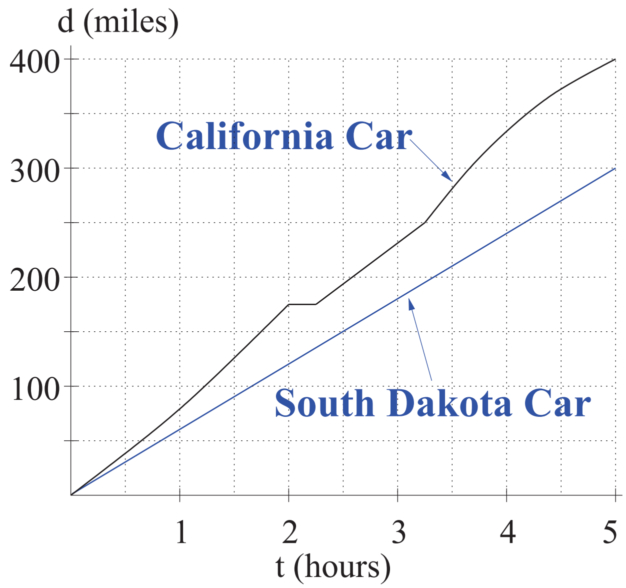
   b. $f(3)$

   $f(3)$ represents the amount of the substance present, in grams, after 3 years.

   c. $\dfrac{f(3)-f(1)}{3-1}$

   This is the average rate at which the substance is decaying, in grams per year, between 1 and 3 years.

3. Two cars travel for 5 hours along Interstate 5. A South Dakotan in a 1983 Chevy Caprice travels 300 miles, always at a constant speed. A Californian in a 2009 Porsche travels 400 miles, but at varying speeds (see graph to the right).



(a) On the axes above, sketch a graph of the distance traveled by the South Dakotan as a function of time.

(b) Compute the average velocity of each car over the 5-hour trip.

For the Chevy Caprice, we have

$$\frac{\Delta d}{\Delta t} = \frac{300 - 0}{5 - 0} = \frac{300 \text{ miles}}{5 \text{ hours}} = 60 \text{ miles per hour},$$

so the Chevy Caprice has an average velocity of 60 miles per hour. For the Porsche, we have

$$\frac{\Delta d}{\Delta t} = \frac{400 - 0}{5 - 0} = \frac{400 \text{ miles}}{5 \text{ hours}} = 80 \text{ miles per hour},$$

so the Porsche has an average velocity of 80 miles per hour.

(c) Does the Californian drive faster than the South Dakotan over the entire 5 hour interval? Justify your answer!

No. For example, the Californian has a velocity of 0 between $t = 2$ hours and $t = 2.25$ hours, because the slope of the graph corresponding to the Porsche is 0 on this time interval.