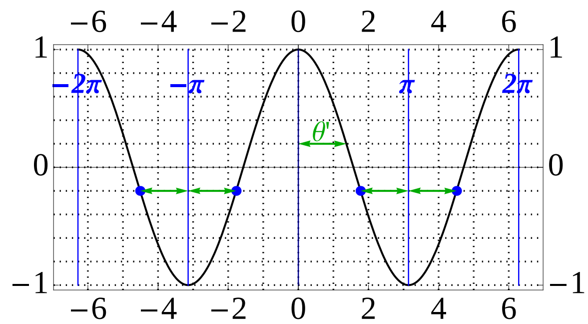
Section 9.1 – Trigonometric Equations

Example 1. Consider the graph of $y = \cos\theta$ given to the right.



(a) Graphically estimate the solutions to $\cos\theta = -0.2$, where $-2\pi \le \theta \le 2\pi$. (**Note:** Angles, θ, are on the horizontal axis)

$-4.5, \quad -1.8, \quad 1.8, \quad 4.5$

(b) Use your calculator to find $\cos^{-1}(-0.2)$, and compare with your answers to (a).

Using radian mode, we have $\cos^{-1}(-0.2) \approx 1.772$, which coincides with one of the four solutions to part (a).

(c) Use your calculator to find $\cos^{-1}(0.2)$, and use this number and symmetry to obtain more accurate solutions to $\cos\theta = -0.2$.

$$\cos^{-1}(0.2) \approx \underline{1.369} \qquad \longleftarrow \qquad \theta'$$

We call the number θ' calculated above the "reference angle" for the equation $\cos x = -0.2$. Referring to the graph above, we see that the four solutions to $\cos x = -0.2$ are:

$$\begin{aligned}
\theta_1 &= -\pi - \theta' &\approx -\pi - 1.369 &\approx \underline{-4.511} \\
\theta_2 &= -\pi + \theta' &\approx -\pi + 1.369 &\approx \underline{-1.772} \\
\theta_3 &= \pi - \theta' &\approx \pi - 1.369 &\approx \underline{1.772} \\
\theta_4 &= -\pi - \theta' &\approx \pi + 1.369 &\approx \underline{4.511}
\end{aligned}$$

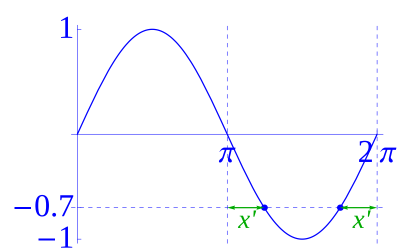
Example 2. For each equation, find all solutions between 0 and 2π, giving exact solutions where possible.

(a) $\sin x = -\frac{7}{10}$ (b) $6\cos x - 1 = 2$

(a) First, we calculate a reference angle x':

$$x' = \sin^{-1}(7/10) \approx 0.775$$

Therefore, using the graph of $y = \sin x$ to the right as a guide, our two solutions are

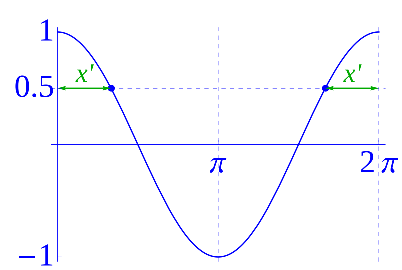


$$\begin{aligned}
x_1 &= \pi + x' &\approx \pi + 0.775 &\approx \underline{3.917} \\
x_2 &= 2\pi - x' &\approx 2\pi - 0.775 &\approx \underline{5.508}
\end{aligned}$$

(b) First, we rewrite the equation as $6\cos x = 3$, and then $\cos x = 1/2$. Our reference angle is therefore

$$x' = \cos^{-1}(1/2) = \pi/3,$$

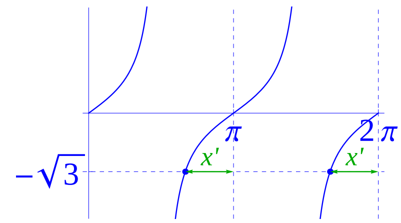
so our two solutions are



$$x_1 = x' = \underline{\frac{\pi}{3}}$$

$$x_2 = 2\pi - \frac{\pi}{3} = \underline{\frac{5\pi}{3}}$$

Example 3. Find all solutions between 0 and 2π to $\tan x = -\sqrt{3}$ exactly.



Consider the graph of $y = \tan x$ given the right. Our reference angle is

$$x' = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3},$$

so our solutions are

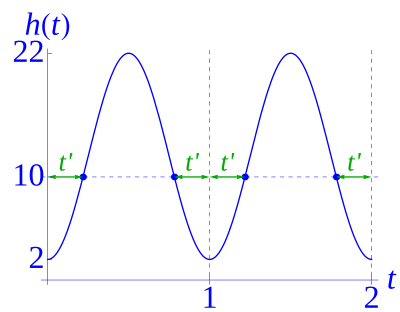
$$x_1 = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \quad \text{and} \quad x_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

Example 4. The height of a rider on a ferris wheel is given by $h(t) = 12 - 10\cos(2\pi t)$ meters, where t gives time, in minutes, after the person boards the ferris wheel.

(a) Find the amplitude, midline, and period of the function h, and then sketch a graph h corresponding to the first two minutes of the ride.

From the formula for h, we see that $B = 2\pi$, $A = -10$, and $k = 12$. Therefore, we have:

$$\begin{aligned} \text{Period} &= \frac{2\pi}{B} = 1 \text{ minute} \\ \text{Amplitude} &= |A| = 10 \text{ meters} \\ \text{Midline} &= k = 12 \text{ meters} \end{aligned}$$



(b) During the first two minutes of the ride, find the times when the rider has a height of 10 meters.

We want to find values of t such that $h(t) = 10$, so we have

$$12 - 10\cos(2\pi t) = 10 \implies -10\cos(2\pi t) = -2 \implies \cos(2\pi t) = 0.2.$$

Next, we find a reference angle as follows:

$$\begin{aligned} \cos(2\pi t') = 0.2 &\implies 2\pi t' = \cos^{-1}(0.2) \\ &\implies t' = \frac{\cos^{-1}(0.2)}{2\pi} \approx 0.218 \end{aligned}$$

Therefore, referring to our graph above, we see that our solutions are given by

$$\begin{aligned} t_1 &= t' \approx 0.218 \\ t_2 &= 1 - t' \approx 0.782 \\ t_3 &= 1 + t' \approx 1.218 \\ t_4 &= 2 - t' \approx 1.782 \end{aligned}$$

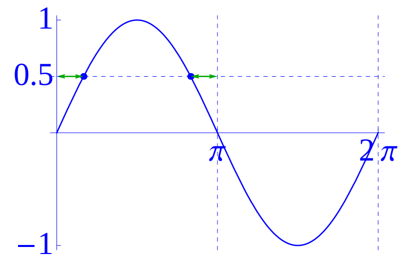
We conclude that the rider is at a height of 10 meters after 0.218, 0.782, 1.218, and 1.782 minutes.

Examples and Exercises

1. Find all solutions between 0 and 2π to each of the following equations, giving exact solutions where possible.

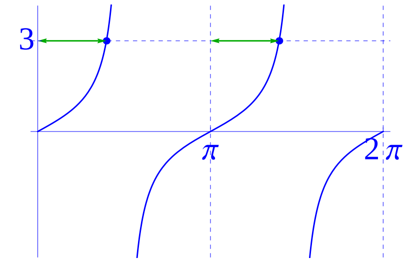
   (a) $\sin x = \frac{1}{2}$ (b) $2\tan x - 5 = 1$ (c) $\cos x = -\frac{\sqrt{2}}{2}$

   (a) First, our reference angle is $x' = \sin^{-1}(1/2) = \pi/6$, so, referring to the graph of $y = \sin x$ shown to the right, our solutions are given by



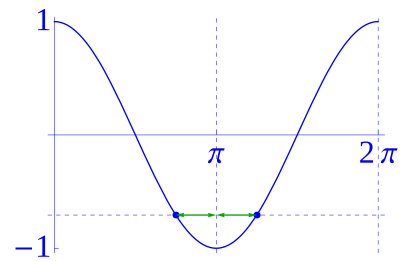
$$x_1 = x' = \underline{\frac{\pi}{6}} \quad \text{and} \quad x_2 = \pi - x' = \underline{\frac{5\pi}{6}}.$$

   (b) First, we simplify our equation as $2\tan x = 6$, and finally as $\tan x = 3$. Therefore, our reference angle is $x' = \tan^{-1} 3 \approx 1.249$. Referring to the graph of $y = \tan x$ given to the right, our solutions are given by



$$x_1 = x' \approx \underline{1.249} \quad \text{and} \quad x_2 = \pi + x' \approx \underline{4.391}.$$

   (c) First, our reference angle is $x' = \cos^{-1}(\sqrt{2}/2) = \pi/4$, so, referring to the graph of $y = \cos x$ shown to the right, our solutions are given by

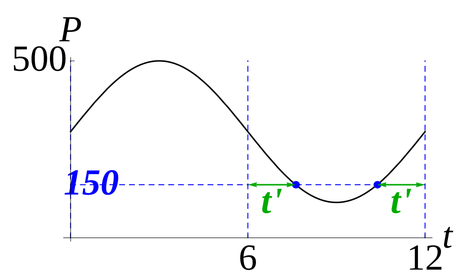


$$x_1 = \pi - x' = \underline{\frac{3\pi}{4}} \quad \text{and} \quad x_2 = \pi + x' = \underline{\frac{5\pi}{4}}.$$

2. An animal population is modeled by the function

$$P = 300 + 200\sin\left(\frac{\pi}{6}t\right)$$

   (see graph to the right), where t represents the time, in days, after the beginning of January 1st. Find the times at which the animal population is 150.



   First, note that

$$P = 150 \implies 300 + 200\sin\left(\frac{\pi}{6}t\right) = 150 \implies \sin\left(\frac{\pi}{6}t\right) = -\frac{150}{200} = -\frac{3}{4},$$

   so our reference angle is given by

$$\sin\left(\frac{\pi}{6}t'\right) = \frac{3}{4} \implies t' = \frac{\sin^{-1}(3/4)}{\pi/6} \approx 1.620$$

   Therefore, using the diagram above as a guide, our solutions are

$$t_1 \approx 6 + 1.620 = \underline{7.62} \quad \text{and} \quad t_2 \approx 12 - 1.620 = \underline{10.38}$$

   Since $t = 7$ corresponds to August and $t = 10$ corresponds to November, this means that the animal population will be 150 once in the latter half of August and again in the first half of November.

Section 9.2 – Identities, Expressions, and Equations

Preliminary Example. Discuss the difference between the equation (a) $\sin\theta = \cos\theta$ and the equation (b) $\sin(2\theta) = 2\sin\theta\cos\theta$

(a) The equation $\sin\theta = \cos\theta$ is only true for some values of θ, like $\theta = \pi/4$ (see table to the right).

(b) The equation $\sin(2\theta) = 2\sin\theta\cos\theta$ is true for all values of θ.

θ	$\cos\theta$	$\sin\theta$	$\sin(2\theta)$	$2\sin\theta\cos\theta$
0	1	0	0	0
$\pi/6$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1
$\pi/3$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$\sqrt{3}/2$
$\pi/2$	0	1	0	0

Note: An equation that is true for all values of the involved variable is called an identity. This means that equation (b) is an identity, but equation (a) is not.

More Trigonometric Identities (continued from Section 7.7)

(8) $\cos^2\theta = 1 - \sin^2\theta$

(9) $\sin^2\theta = 1 - \cos^2\theta$

(10) $\sec^2\theta = \tan^2\theta + 1$

(11) $\tan(-\theta) = -\tan\theta$

(12) $\sin(2\theta) = 2\sin\theta\cos\theta$

(13) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

(14) $\cos(2\theta) = 1 - 2\sin^2\theta$

(15) $\cos(2\theta) = 2\cos^2\theta - 1$

Example 1.

(a) Use appropriate identities from Section 8.3 to prove Identity (11) above.

We will begin with the left-hand side of (11) and show that it equals the right-hand side. We have

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} \quad \longleftarrow \quad \text{using (3) and (4) on from Section 8.3}$$

$$= -\tan\theta$$

(b) Divide both sides of Identity (9) by $\cos^2\theta$ and simplify to obtain Identity (10).

We have

$$\sin^2\theta = 1 - \cos^2\theta \implies \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta}$$

$$\implies \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta}$$

$$\implies \tan^2\theta = \sec^2\theta - 1$$

Therefore, by adding 1 to both sides, we obtain $\sec^2\theta = \tan^2\theta + 1$.

Example 2. Are any of the following identities? If so, prove them algebraically.

(a) $\cos\left(\frac{1}{x}\right) = \frac{1}{\cos x}$ (b) $2\tan x\cos^2 x = \sin(2x)$

(a) This is not an identity. To see that this is the case, let $x = 1$, and note that

$$\cos\left(\frac{1}{1}\right) = \cos 1 \approx 0.540$$
$$\frac{1}{\cos 1} \approx \frac{1}{0.540} \approx 1.851$$

Therefore, since $0.540 \neq 1.851$, the equation is not true for $x = 1$, so it is not an identity.

(b) This is an identity because

$$2\tan x\cos^2 x = 2\cdot\frac{\sin x}{\cos x}\cdot\frac{\cos^2 x}{1} = \frac{2\sin x\cos^2 x}{\cos x} = 2\sin x\cos x = \sin(2x).$$

Example 3. Rewrite the expression $\sec\theta - \cos\theta$ so that your final answer is a product of two trig functions.

We have

$$\sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta = \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta} = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} = \frac{\sin\theta}{1}\cdot\frac{\sin\theta}{\cos\theta} = \sin\theta\,\tan\theta.$$

Therefore, our final answer is $\sin\theta\,\tan\theta$.

Example 4. Solve the equation $2\sin^2\theta = 3\cos\theta + 3$ for $0 \le \theta \le 2\pi$.

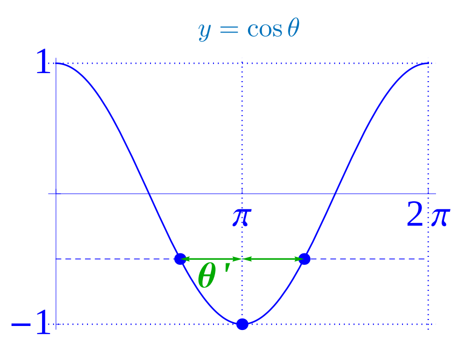
Using the identity $\sin^2\theta = 1 - \cos^2\theta$, we have

$$2\sin^2\theta = 3\cos\theta + 3 \implies 2(1-\cos^2\theta) = 3\cos\theta + 3 \implies 2 - 2\cos^2\theta = 3\cos\theta + 3$$
$$\implies 2\cos^2\theta + 3\cos\theta + 1 = 0$$
$$\implies (2\cos\theta + 1)(\cos\theta + 1) = 0$$
$$\implies \cos\theta = -\frac{1}{2} \text{ or } \cos\theta = -1.$$

Since π radians is the only angle between 0 and 2π whose cosine equals -1, we see that $\theta_1 = \pi$ is the only solution to the equation $\cos\theta = -1$.

To finish, we must solve the equation $\cos\theta = -1/2$. Our reference angle is $\theta' = \cos^{-1}(1/2) = \pi/3$, so, referring to the graph to the right, our solutions are:

$$\theta_2 = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$
$$\theta_3 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$



Combining our work above, we see that our final answer is $\left\{\pi, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

Examples and Exercises

1. (a) Simplify $\sin A(\csc A - \sin A)$ so that your final answer is a single trig function raised to a power.

$$\begin{aligned}\sin A(\csc A - \sin A) &= \sin A \csc A - \sin^2 A = \frac{\sin A}{1}\cdot\frac{1}{\sin A} - \sin^2 A = 1 - \sin^2 A \\ &= \underline{\cos^2 A}.\end{aligned}$$

(b) Simplify $\dfrac{1-\cos^2\theta}{\cos\theta}$ so that your answer is a product of two trig functions with no fractions.

$$\frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta} = \frac{\sin\theta}{1}\cdot\frac{\sin\theta}{\cos\theta} = \underline{\sin\theta\tan\theta}.$$

(c) Simplify $\dfrac{\cos(2t)}{\cos t + \sin t}$ so that your answer is the difference between two trig functions with no fractions. (**Hint:** Use Formula 13 from Section 8.3 and then factor the numerator.)

$$\frac{\cos(2t)}{\cos t + \sin t} = \frac{\cos^2 t - \sin^2 t}{\cos t + \sin t} = \frac{(\cos t - \sin t)(\cos t + \sin t)}{\cos t + \sin t} = \underline{\cos t - \sin t}.$$

(d) Simplify $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta}$ so that your answer is a constant multiple of one trig function, with no fractions.

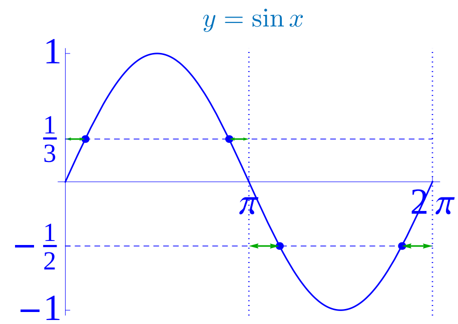
$$\begin{aligned}\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} &= \frac{\sin^2\theta}{(1+\cos\theta)\sin\theta} + \frac{(1+\cos\theta)^2}{(1+\cos\theta)\sin\theta} = \frac{\sin^2\theta + (1+2\cos\theta+\cos^2\theta)}{(1+\cos\theta)\sin\theta} \\ &= \frac{(\sin^2\theta+\cos^2\theta)+1+2\cos\theta}{(1+\cos\theta)\sin\theta} \\ &= \frac{2+2\cos\theta}{(1+\cos\theta)\sin\theta} \\ &= \frac{2(1+\cos\theta)}{(1+\cos\theta)\sin\theta} \\ &= \frac{2}{\sin\theta} \\ &= \underline{2\csc\theta}\end{aligned}$$

2. Find all solutions to $5\sin^2 x + \sin x = \cos^2 x$ that lie in the interval $0 \le x < 2\pi$.

Using the identity $\cos^2 x = 1 - \sin^2 x$, we have

$$5\sin^2 x + \sin x = \cos^2 x \implies 5\sin^2 x + \sin x = 1 - \sin^2 x \implies 6\sin^2 x + \sin x - 1 = 0$$
$$\implies (3\sin x - 1)(2\sin x + 1) = 0$$
$$\implies \sin x = \frac{1}{3} \quad \text{or} \quad \sin x = -\frac{1}{2}$$

For the first equation, $\sin x = 1/3$, our reference angle is $x' = \sin^{-1}(1/3) \approx 0.340$. Therefore, referring to the graph, we obtain two solutions:



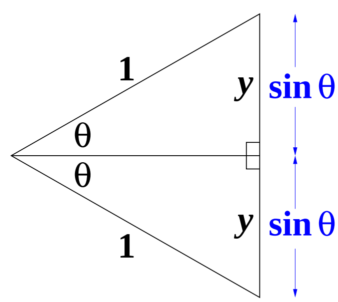
$$x_1 = x' \approx 0.340$$
$$x_2 = \pi - 0.340 \approx 2.802$$

Similarly, for the second equation, our reference angle is $x' = \sin^{-1}(1/2) = \pi/6$, so, we obtain two more solutions:

$$x_3 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$
$$x_4 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Combining our work above, we see that our final answer is $\left\{0.340,\ 2.802,\ \frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

3. Consider the diagram to the right.



(a) Use right triangle trigonometry to find an expression for y in terms of θ.

Referring to either of the two right triangles in the diagram, we have

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{y}{1},$$

so $y = \sin\theta$, which is our answer to (a).

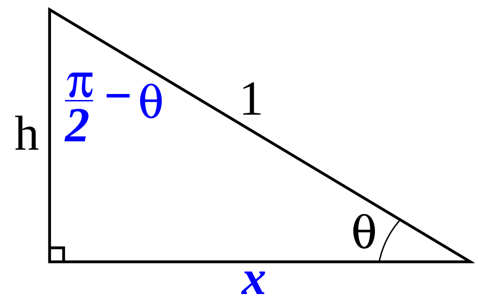
(b) Prove the identity $\cos(2\theta) = 1 - 2\sin^2\theta$. Do this by applying the Law of Cosines to the "outer" triangle in the diagram; that is, the triangle with sides of length 1, 1, and $2y$.

First, note that the vertex angle opposite the side of length $2y$ in the outer triangle is 2θ. Therefore, by the Law of Cosines, we have

$$(2y)^2 = 1^2 + 1^2 - 2(1)(1)\cos(2\theta) \implies 4y^2 = 2 - 2\cos(2\theta)$$
$$\implies 4\sin^2\theta = 2\big(1 - \cos(2\theta)\big)$$
$$\implies 2\sin^2\theta = 1 - \cos(2\theta)$$
$$\implies 2\sin^2\theta + \cos(2\theta) = 1$$
$$\implies \cos(2\theta) = 1 - 2\sin^2\theta$$

4. Given the right triangle to the right, write each of the following quantities in terms of h.

   Note. Asking you to "write something in terms of h" is NOT asking you to solve for h. It is asking you to rewrite the quantity that you are given so that h is the only unknown in your answer.



$$x^2 + h^2 = 1^2 \implies x^2 = 1 - h^2 \implies x = \sqrt{1 - h^2}$$

(a) $\sin\theta$

$$\sin\theta = \frac{\texttt{OPP}}{\texttt{HYP}} = \frac{h}{1},$$

so our answer is h.

(b) $\cos\theta$

$$\cos\theta = \frac{\texttt{ADJ}}{\texttt{HYP}} = \frac{x}{1} = \frac{\sqrt{1-h^2}}{1}$$

so our answer is $\sqrt{1-h^2}$.

(c) $\cos(\frac{\pi}{2} - \theta)$

$$\cos(\frac{\pi}{2} - \theta) = \frac{\texttt{ADJ}}{\texttt{HYP}} = \frac{h}{1},$$

so our answer is h.

(d) $\sin(2\theta)$

$$\sin(2\theta) = 2\sin\theta\cos\theta = 2(h)(\sqrt{1-h^2}),$$

so our answer is $2h\sqrt{1-h^2}$.

(e) $\cos(\sin^{-1} h)$

By definition, $\sin^{-1} h$ is the angle whose sine equals h. By part (a), this means that $\sin^{-1} h = \theta$. Therefore, we have

$$\cos(\sin^{-1} h) = \cos\theta = \sqrt{1-h^2}, \qquad \text{(by part (b))}$$

so our answer is $\sqrt{1-h^2}$.