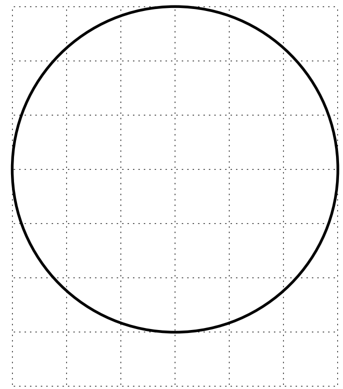
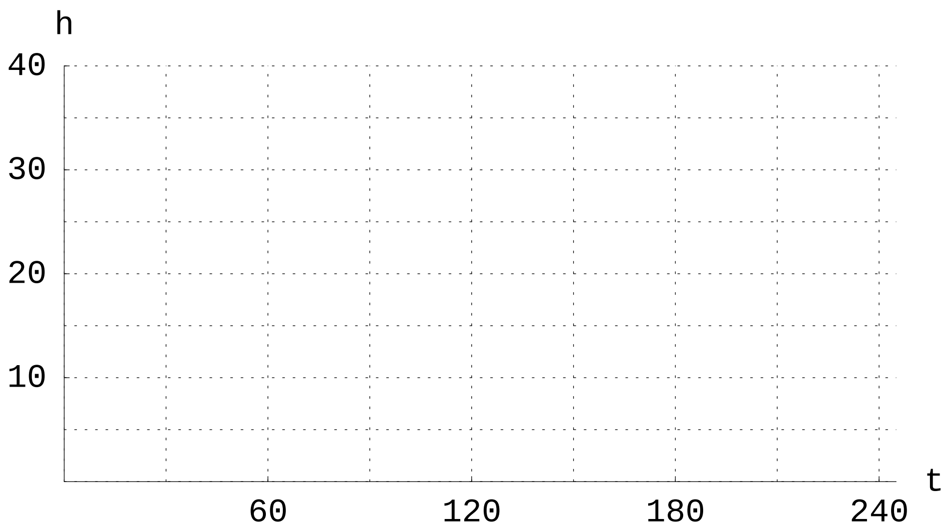
Section 7.1 – Introduction to Periodic Functions

Preliminary Example. The Brown County Ferris Wheel has diameter 30 meters and completes one full revolution every two minutes. When you are at the lowest point on the wheel, you are still 5 meters above the ground. Assuming you board the ride at $t = 0$ seconds, sketch a graph of your height, $h = f(t)$, as a function of time.





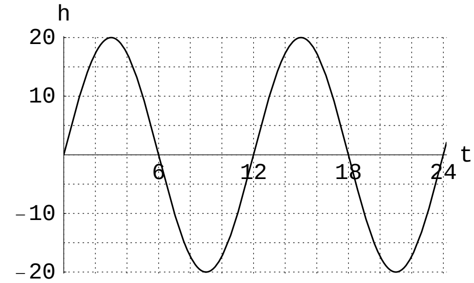
Definition. A function f is called *periodic* if its output values repeat at regular intervals. Graphically, this means that if the graph of f is shifted horizontally by p units, the new graph is identical to the original. Given a periodic function f:

1. The *period* is the horizontal distance that it takes for the graph to complete one full cycle. That is, if p is the period, then $f(t + p) = f(t)$.
2. The *midline* is the horizontal line midway between the function's maximum and minimum output values.
3. The *amplitude* is the vertical distance between the function's maximum value and the midline.

Example 1. What are the amplitude, midline and period of the function $h = f(t)$ from the preliminary example?

Examples and Exercises

1. The function given below models the height, h, in feet, of the tide above (or below) mean sea level t hours after midnight.



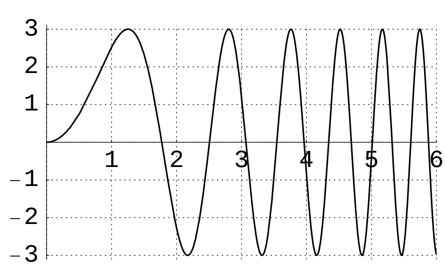
   (a) Is the tide rising or falling at 1:00 a.m.?

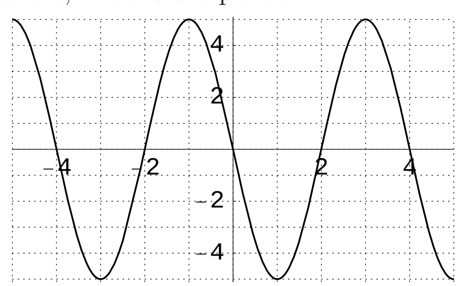
   (b) When does low tide occur?

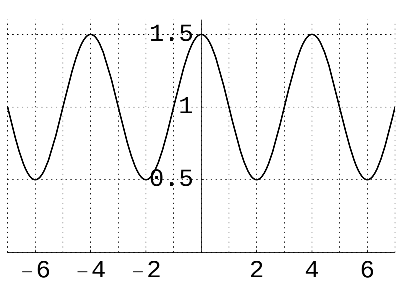
   (c) What is the amplitude of the function? Give a practical interpretation of your answer.

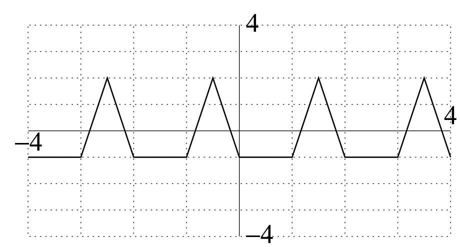
   (d) What is the midline of the function? Give a practical interpretation of your answer.

2. Which of the following functions are periodic? For those that are, what is the period?





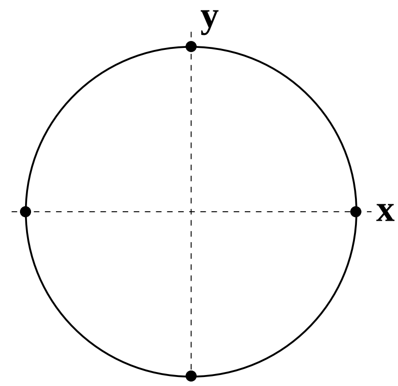




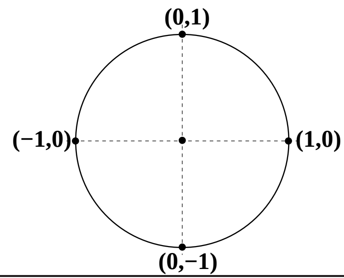
Section 7.2 – The Sine and Cosine Functions

Angle Measurement in Circles

- Angles start from the positive x-axis.
- Counterclockwise defined to be positive.



Definition. The *unit circle* is the term used to describe a circle that has its center at the origin and has radius equal to 1. The *cosine* and *sine* functions are then defined as described below.

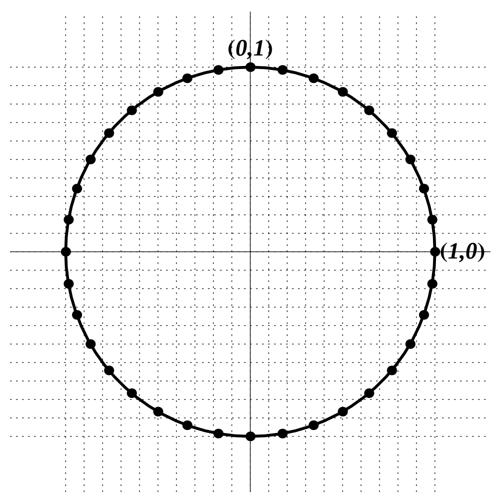


Example 1. On the unit circle to the right, the angles 10°, 20°, 30°, etc., are indicated by black dots on the circle. Use this diagram to estimate each of the following:

(a) $\cos(30^\circ) =$ ______

(b) $\sin(150^\circ) =$ ______

(c) $\cos(270^\circ) =$ ______

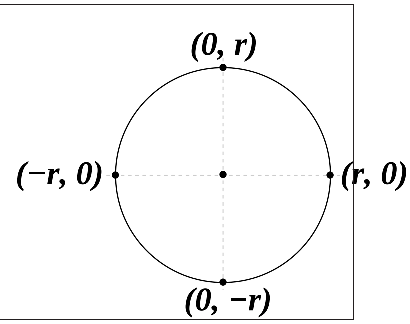


Example 2.

(a) Find an angle θ between 0° and 360° that has the same sine as 40°.

(b) Find an angle θ between 0° and 360° that has the same cosine as 40°.

Theorem. Consider a circle of radius r centered at the origin. Then the x and y coordinates of a point on this circle are given by the following formulas:



Examples and Exercises

1. Use the unit circle to the right to estimate each of the following quantities to the nearest 0.05 of a unit.

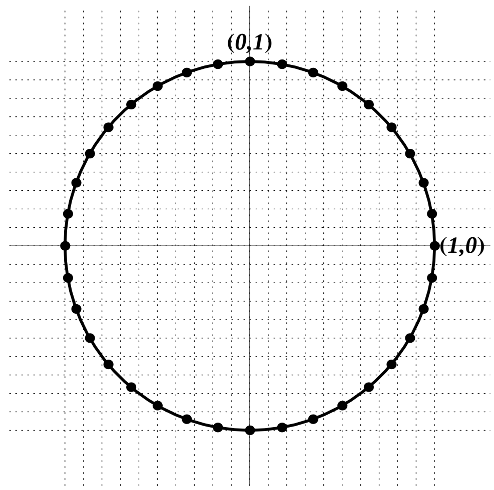
   (a) $\sin(90°) =$ ______ (b) $\cos(90°) =$ ______

   (c) $\sin(180°) =$ ______ (d) $\cos(180°) =$ ______

   (e) $\cos(45°) =$ ______ (f) $\sin(-90°) =$ ______

   (g) $\cos(70°) =$ ______ (h) $\sin(190°) =$ ______

   (i) $\sin(110°) =$ ______ (j) $\cos(110°) =$ ______



2. For each of the following, fill in the blank with an angle between 0° and 360°, different from the first one, that makes the statement true.

   (a) $\sin(20°) = \sin($______$)$ (b) $\sin(70°) = \sin($______$)$ (c) $\sin(225°) = \sin($______$)$

   (d) $\cos(20°) = \cos($______$)$ (e) $\cos(70°) = \cos($______$)$ (f) $\cos(225°) = \cos($______$)$

3. Given to the right is a unit circle. Fill in the blanks with the correct answer in terms of a or b.

   (a) $\sin(\theta + 360°) =$ ______

   (b) $\sin(\theta + 180°) =$ ______

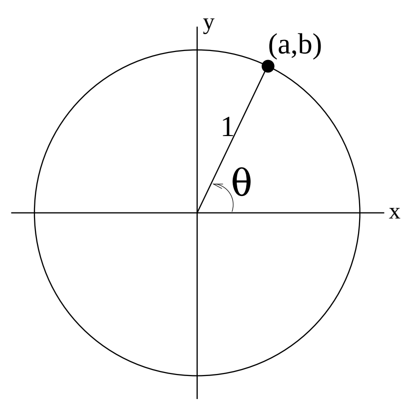
   (c) $\cos(180° - \theta) =$ ______

   (d) $\sin(180° - \theta) =$ ______

   (e) $\cos(360° - \theta) =$ ______

   (f) $\sin(360° - \theta) =$ ______

   (g) $\sin(90° - \theta) =$ ______

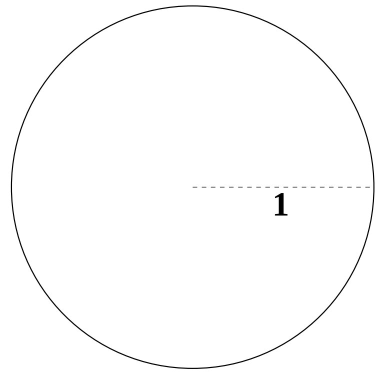


4. Use your calculator to find the coordinates of the point P at the given angle on a circle of radius 4 centered at the origin.

   (a) $70°$ (b) $255°$

Section 7.3 – Radians and Arc Length

Definition. An angle of 1 radian is defined to be the angle, in the counterclockwise direction, at the center of a unit circle which spans an arc of length 1.



Conversion Factors:

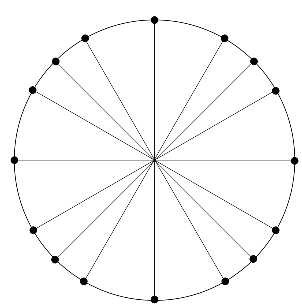
Degrees × $\longrightarrow$ Radians

Radians × $\longrightarrow$ Degrees

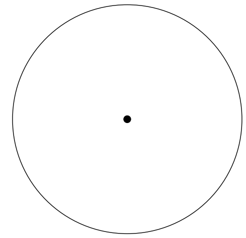
Example 2. Convert each of the following angles from radians to degrees or from degrees to radians. An angle measure is assumed to be in radians if the degree symbol is not indicated after it.

(a) $30°$ (b) $\frac{3\pi}{2}$ (c) 1.4

Example 3. On the unit circle to the right, label the indicated "common" angles with their degree and radian measures.

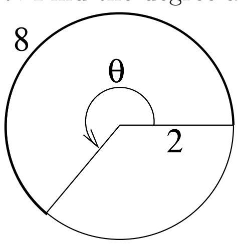


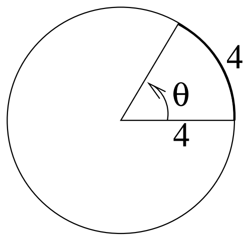
Theorem. The arc length, s, spanned in a circle of radius r by an angle of θ radians, $0 \le \theta \le 2\pi$, is given by



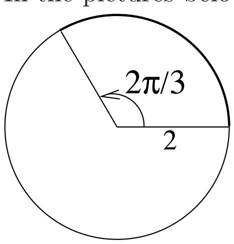
Examples and Exercises

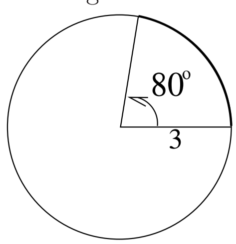
1. In the pictures below, you are given the radius of a circle and the length of a circular arc cut off by an angle θ. Find the degree and radian measure of θ.





2. In the pictures below, find the length of the arc cut off by each angle.





3. A satellite orbiting the earth in a circular path stays at a constant altitude of 100 kilometers throughout its orbit. Given that the radius of the earth is 6370 kilometers, find the distance that the satellite travels in completing 70% of one complete orbit.

4. An ant starts at the point (0,3) on a circle of radius 3 (centered at the origin) and walks 2 units counterclockwise along the arc of the circle. Find the x and the y coordinates of where the ant ends up.

Section 7.3 Supplement

Preliminary Example. Use the unit circles and corresponding triangles below to find the exact value of the sine and cosine of the special angles $30°$, $45°$, and $60°$.

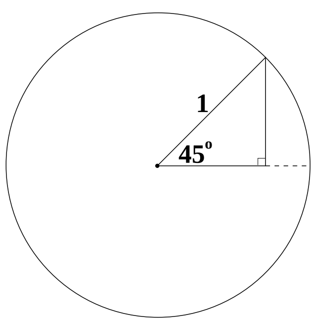


Figure 1.

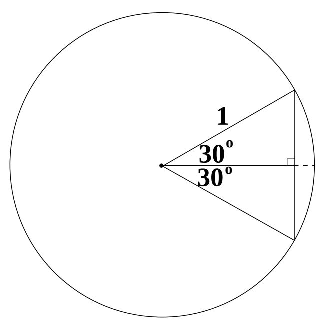


Figure 2.

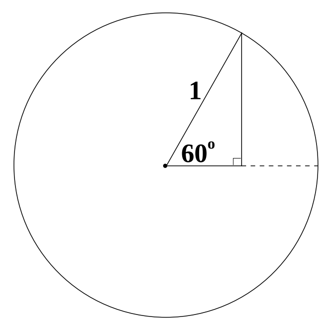
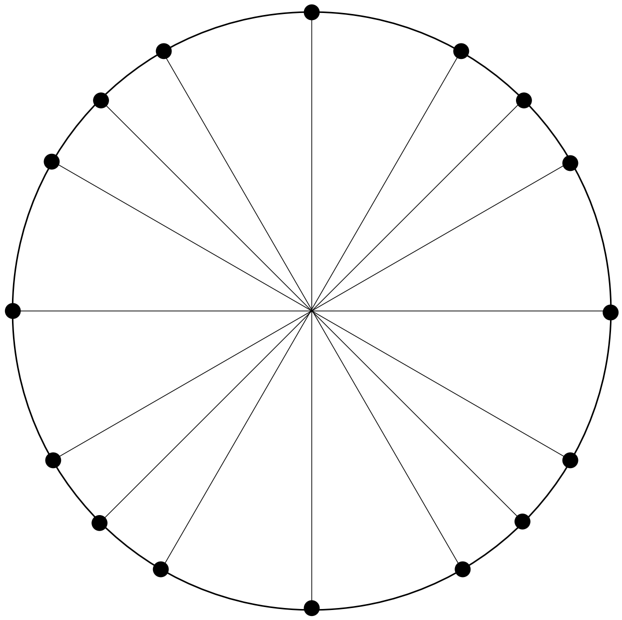


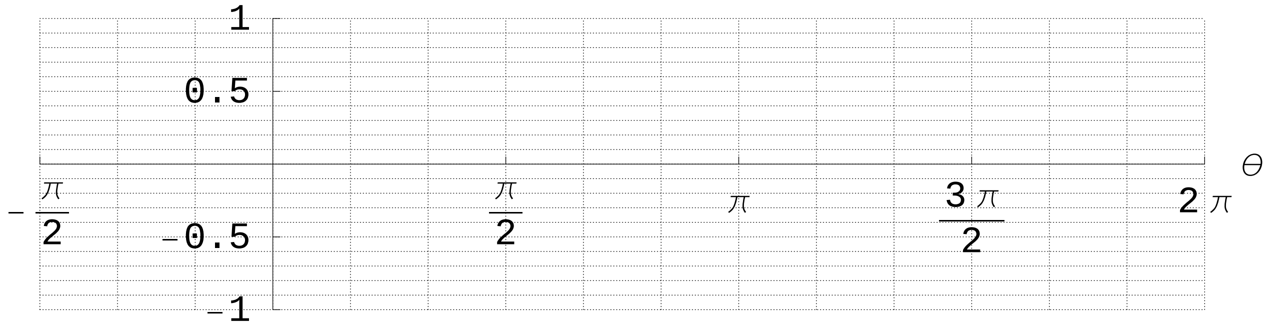
Figure 3.

The Unit Circle



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
θ	$0°$	$30°$	$45°$	$60°$	$90°$	$120°$	$135°$	$150°$	$180°$
$\cos\theta$									
$\sin\theta$									

θ	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
θ	$210°$	$225°$	$240°$	$270°$	$300°$	$315°$	$330°$	$360°$
$\cos\theta$								
$\sin\theta$								



Sections 7.4 and 7.5 – Sinusoidal Functions

Directions. Make sure that your graphing calculator is set in radian mode.

Function	Effect on $y = \sin x$
$y = 2\sin x$	
$y = \sin x + 2$	
$y = \sin(x + 2)$	
$y = \sin(2x)$	

B	$y = \sin(Bx)$	Period
1	$y = \sin x$	
2	$y = \sin(2x)$	
4	$y = \sin(4x)$	
1/2	$y = \sin(x/2)$	
B	$y = \sin(Bx)$	

Summary

For the sinusoidal functions $y = A\sin(B(x - h)) + k$ and $y = A\cos(B(x - h)) + k$:

1. Amplitude = ______________
2. Period = ______________
3. Horizontal Shift = ______________
4. Midline: ______________

Definition. A function is called *sinusoidal* if it is a transformation of a sine or a cosine function.

Primary Goal in Section 7.5. Find formulas for sinusoidal functions given graphs, tables, or verbal descriptions of the functions.

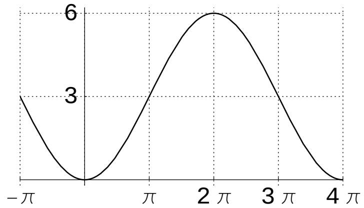
Helpful Hints in Finding Formulas for Sinusoidal Functions

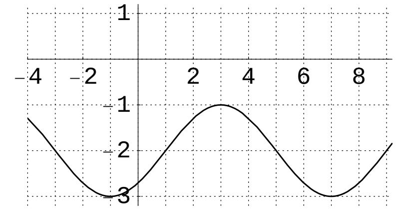
1. If selected starting point occurs at the midline of the graph, use the sine function.
2. If selected starting point occurs at the maximum or minimum value of the graph, use the cosine function.
3. Changing the sign of the constant "A" reflects the graph of a sinusoidal function about its midline.

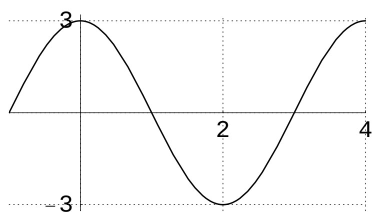
Example 1. Let $y = 2\sin(2x - \pi) + 2$. Find the amplitude, period, midline, and horizontal shift of this function.

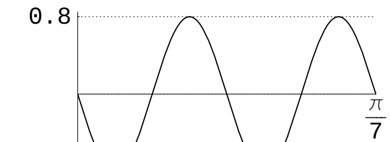
Examples and Exercises

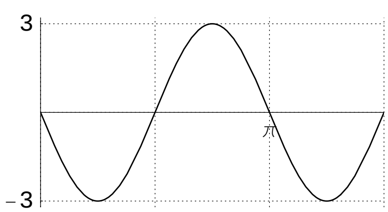
1. Find a possible formula for each of the following sinusoidal functions.

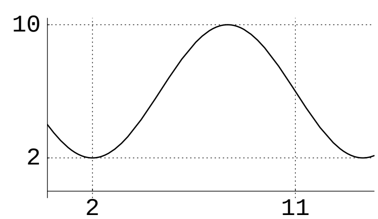












2. For each of the following, find the amplitude, the period, the horizontal shift, and the midline.

   (a) $y = 2\cos(\pi x + \frac{2\pi}{3}) - 1$

   (b) $y = 3 - \sin(2x - 7\pi)$

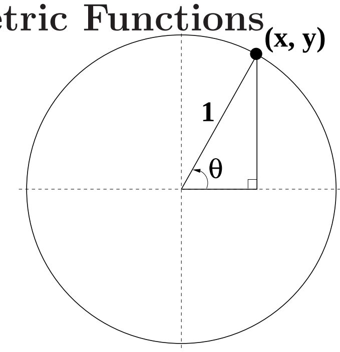
3. A population of animals oscillates annually from a low of 1300 on January 1st to a high of 2200 on July 1st, and back to a low of 1300 on the following January. Assume that the population is well-approximated by a sine or a cosine function.

   (a) Find a formula for the population, P, as a function of time, t. Let t represent the number of months after January 1st. (**Hint.** First, make a rough sketch of the population, and use the sketch to find the amplitude, period, and midline.)

   (b) Estimate the animal population on May 15th.

   (c) On what dates will the animal population be halfway between the maximum and the minimum populations?

Section 7.6 – Tangent and Other Trigonometric Functions



$\cos\theta =$

$\sin\theta =$

$\tan\theta =$ $\qquad\qquad$ $\cot\theta =$

$\sec\theta =$ $\qquad\qquad$ $\csc\theta =$

Example 1. Suppose that $\cos\theta = \frac{2}{5}$ and that θ is in the 4th quadrant. Find $\sin\theta$ and $\tan\theta$ exactly.

Example 2. Find exact values for each of the following:

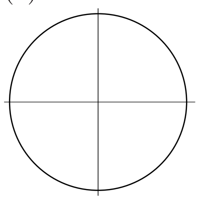
(a) $\tan\left(\frac{\pi}{6}\right)$ $\qquad$ (b) $\tan\left(\frac{\pi}{4}\right)$ $\qquad$ (c) $\tan\left(\frac{\pi}{2}\right)$

Reference Angles

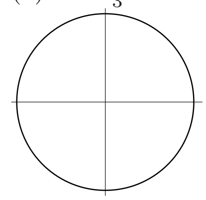
Definition. The *reference angle* associated with an angle θ is the acute angle (having positive measure) formed by the x-axis and the terminal side of the angle θ.

Example. For each of the following angles, sketch the angle and find the reference angle.

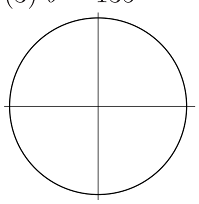
(1) $\theta = 300^\circ$



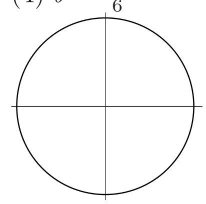
(2) $\theta = \frac{4\pi}{3}$



(3) $\theta = 135^\circ$



(4) $\theta = \frac{7\pi}{6}$



Key Fact. If θ is any angle and θ' is the reference angle, then

$$\begin{aligned} \sin\theta' &= \pm\sin\theta & \tan\theta' &= \pm\tan\theta & \sec\theta' &= \pm\sec\theta \\ \cos\theta' &= \pm\cos\theta & \csc\theta' &= \pm\csc\theta & \cot\theta' &= \pm\cot\theta, \end{aligned}$$

where the correct sign must be chosen based on the quadrant of the angle θ.

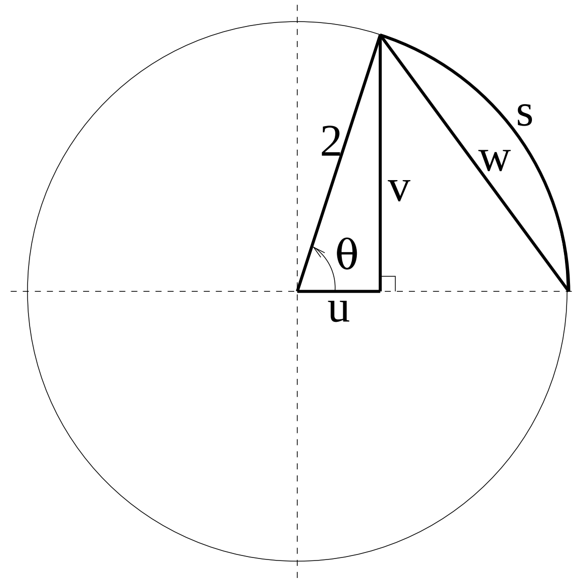
Exercise. Return to the previous example and find the <u>exact</u> value of the sine and the cosine of each angle.

Examples and Exercises

1. Suppose that $\sin\theta = -\frac{3}{4}$ and that $\frac{3\pi}{2} \leq \theta \leq 2\pi$. Find the **exact values** of $\cos\theta$ and $\sec\theta$.

2. Suppose that $\csc\theta = \frac{x}{2}$ and that θ lies in the 2nd quadrant. Find expressions for $\cos\theta$ and $\tan\theta$ in terms of x.

3. Given to the right is a circle of radius 2 feet (not drawn to scale). The length of the circular arc s is 2.6 feet. Find the lengths of the segments labeled u, v, and w. Give all answers rounded to the nearest 0.001.



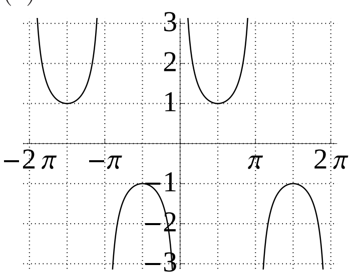
Section 7.7 – Trigonometric Relationships and Identities

Definitions. We define the *secant*, *cosecant*, and *cotangent* functions as follows:

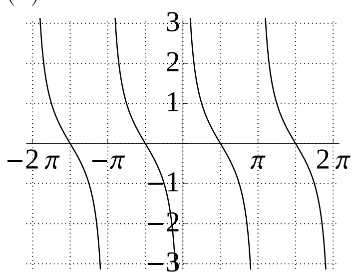
$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Preliminary Exercise. Shown below are graphs of $y = \sec\theta$, $y = \csc\theta$, and $y = \cot\theta$ (not necessarily in that order). Without using a graphing calculator, match each graph to the correct formula. (**Suggestion:** Use your knowledge of the behavior of the sine and cosine functions to make your matches; in particular, pay attention to where these functions are positive, negative, or zero.)

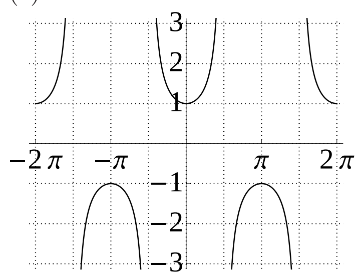
(a)



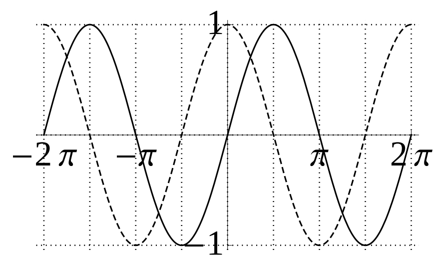
(b)



(c)



Example 1. Given to the right are graphs of the sine and cosine functions. First, label which is which and then complete the following:



(a) Describe how the cosine function can be translated to obtain the sine function, and vice versa. Then, use your observations to complete identities (1) and (2) below.

(b) Is the cosine function even, odd, or neither? How about the sine function? Use your observations to complete identities (3) and (4) below.

Identities.

(1) $\cos\left(\underline{\qquad\qquad}\right) = \sin\theta$

(2) $\sin\left(\underline{\qquad\qquad}\right) = \cos\theta$

(3) $\cos(-\theta) = \underline{\qquad\qquad}$

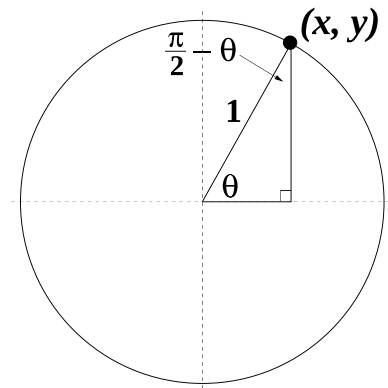
(4) $\sin(-\theta) = \underline{\qquad\qquad}$

Example 2. Illustrate Identity (2) from page **??** by filling in the table to the right. What do you notice?

θ	$\cos\theta$	$\sin\left(\theta+\frac{\pi}{2}\right)$
0		
$\pi/6$		
$\pi/4$		
$\pi/3$		
$\pi/2$		

More Identities

Some observations:



Identities.

(5) $\sin^2\theta+\cos^2\theta \;=\; 1$ (6) $\cos\left(\frac{\pi}{2}-\theta\right) \;=\; \sin\theta$ (7) $\sin\left(\frac{\pi}{2}-\theta\right) \;=\; \cos\theta$

Example 3. Given that $\sin\theta=\frac{1}{3}$ and that θ is in the second quadrant, find the exact values of $\cos\theta$, $\tan\theta$, and $\sec\theta$.

Examples and Exercises

1. Suppose that $\sin\theta = -\frac{3}{4}$ and that $\frac{3\pi}{2} \leq \theta \leq 2\pi$. Find the **exact values** of $\cos\theta$ and $\sec\theta$.

2. Suppose that $\csc\theta = \frac{x}{2}$ and that θ lies in the 2nd quadrant. Find expressions for $\cos\theta$ and $\tan\theta$ in terms of x.

Section 7.8 – Inverse Trigonometric Functions

Preliminary Idea.

$\sin(\pi/6) = 1/2$ means the same thing as ______________________________.

Definition.

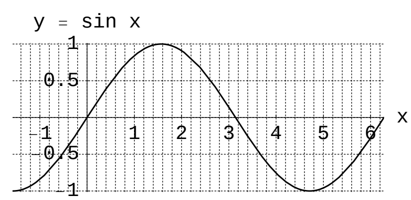
1. $\sin^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is x.
2. $\tan^{-1} x$ is the angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.
3. $\cos^{-1} x$ is the angle between 0 and π whose cosine is x.

Note. "$\sin^{-1} x$", "$\cos^{-1} x$", and "$\tan^{-1} x$" can also be written as "$\arcsin x$", "$\arccos x$", and "$\arctan x$", respectively.

Example 1. Calculate each of the following exactly.

1. $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) =$ ________
2. $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) =$ ________
3. $\tan^{-1}(\sqrt{3}) =$ ________
4. $\sin^{-1}(-1) =$ ________

Example 2. Use the graph to estimate, to the nearest 0.1, all solutions to the equation $\sin x = -\frac{1}{2}$ that lie between 0 and 2π. Then, find the solutions exactly.



Examples and Exercises

1. Solve each of the following trigonometric equations, giving all solutions between 0 and 2π. Give **exact** answers whenever possible.

   (a) $\sin\theta = \frac{\sqrt{3}}{2}$

   (b) $\tan\theta = -0.3$

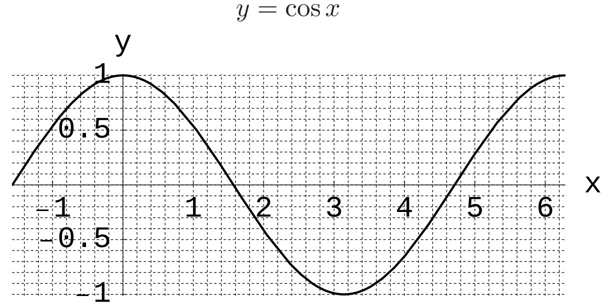
   (c) $\cos\theta = -\frac{1}{2}$

   (d) $\sin\theta = 0.7$

2. Find all solutions to $2 \sin x \cos x + \cos x = 0$ that lie between 0 and 2π. Give your answers **exactly**.

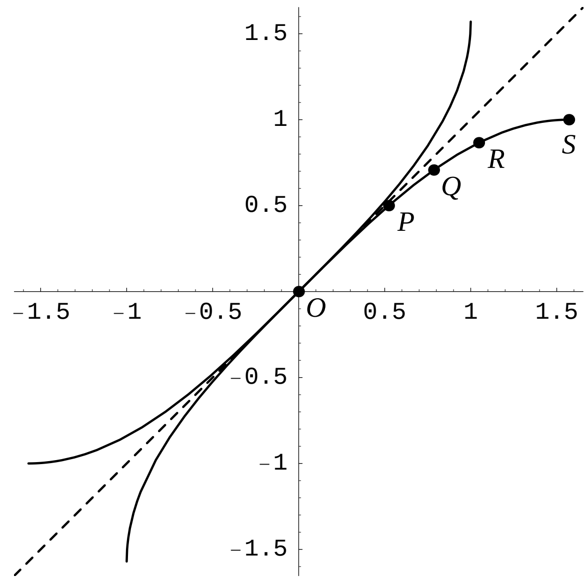
3. Use the graph to the right to estimate the solutions to the equation $\cos x = 0.8$ that lie between 0 and 2π. Then, use reference angles to find more accurate estimates of your solutions.

$y = \cos x$



x	$\sin x$
0	0
$\pi/6$	$1/2$
$\pi/4$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$
$\pi/2$	1

x	$\sin^{-1} x$



x	$\tan x$
0	0
$\pi/6$	$\sqrt{3}/3$
$\pi/4$	1
$\pi/3$	$\sqrt{3}$
$\pi/2$	undefined

x	$\tan^{-1} x$