

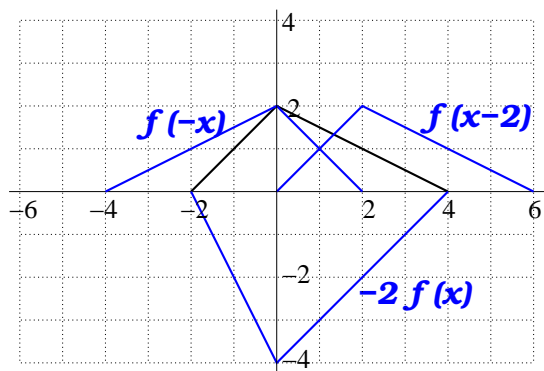
Sections 6.1-6.5 – Function Transformations

Example 1. Consider the function $f(x) = x^2 - 4x + 4$.

Transformation	Formula	Graph	Description
$y = f(x) + 2$	$y = (x^2 - 4x + 4) + 2$ $= x^2 - 4x + 6$		shift up 2
$y = f(x) - 2$	$y = (x^2 - 4x + 4) - 2$ $= x^2 - 4x + 2$		shift down 2
$y = f(x + 2)$	$y = (x + 2)^2 - 4(x + 2) + 4$ $= (x^2 + 4x + 4) - 4x - 8 + 4$ $= x^2$		shift left 2
$y = f(x - 2)$	$y = (x - 2)^2 - 4(x - 2) + 4$ $= (x^2 - 4x + 4) - 4x + 8 + 4$ $= x^2 - 8x + 16$		shift right 2
$y = f(-x)$	$y = (-x)^2 - 4(-x) + 4$ $= x^2 + 4x + 4$		reflection about y-axis
$y = -f(x)$	$y = -(x^2 - 4x + 4)$ $= -x^2 + 4x - 4$		reflection about x-axis

Example 2. Let $y = f(x)$ be the function whose graph is given to the right. Sketch the graphs of the transformations $y = f(x - 2)$, $y = -2f(x)$, and $y = f(-x)$. Then, fill in the entries in the table below.

x	-4	-2	0	2	4	6
$f(x)$	--	0	2	1	0	--
$f(x - 2)$	--	--	0	2	1	0
$-2f(x)$	--	0	-4	-2	0	--
$f(-x)$	0	1	2	0	--	--



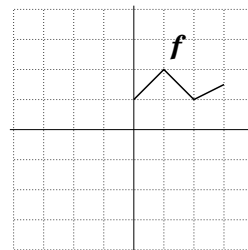
Note: We can also figure out the entries in the table algebraically. For example, we calculate as follows for $y = f(x - 2)$:

$$\begin{aligned} x = 0 &\implies y = f(x - 2) = f(0 - 2) = f(-2) = 0 \\ x = 2 &\implies y = f(x - 2) = f(2 - 2) = f(0) = 2 \\ x = 4 &\implies y = f(x - 2) = f(4 - 2) = f(2) = 1 \\ x = 6 &\implies y = f(x - 2) = f(6 - 2) = f(4) = 0 \end{aligned}$$

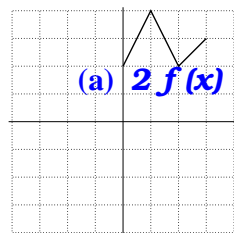
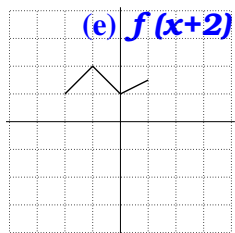
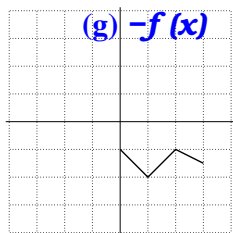
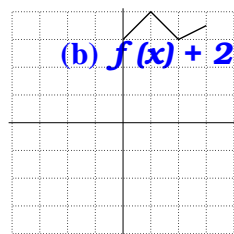
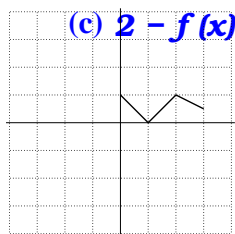
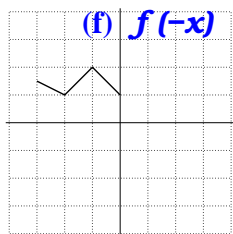
Note that the domain of f is $-2 \leq x \leq 4$, but the domain of $f(x - 2)$ is $0 \leq x \leq 6$ because of the right shift. Similarly, for $-2f(x)$, we have:

$$\begin{aligned} x = -2 &\implies y = -2f(x) = -2f(-2) = -2 \cdot 0 = 0 \\ x = 0 &\implies y = -2f(x) = -2f(0) = -2 \cdot 2 = -4 \\ x = 2 &\implies y = -2f(x) = -2f(2) = -2 \cdot 1 = -2 \\ x = 4 &\implies y = -2f(x) = -2f(4) = -2 \cdot 0 = 0 \end{aligned}$$

Example 3. To the right, you are given the graph of a function f . Match each graph below to the appropriate transformation formula. Note that some transformation formulas will not match any of the graphs.

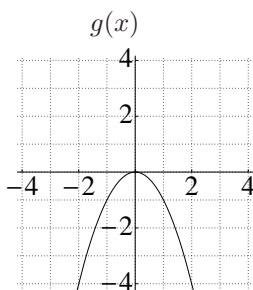
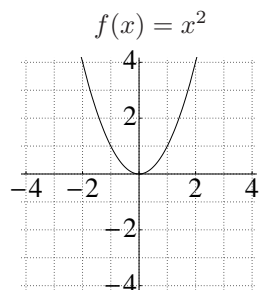


- (a) $y = 2f(x)$ (d) $y = 2f(x) + 2$ (g) $y = -f(x)$
 (b) $y = f(x) + 2$ (e) $y = f(x + 2)$ (h) $y = 4f(x)$
 (c) $y = 2 - f(x)$ (f) $y = f(-x)$



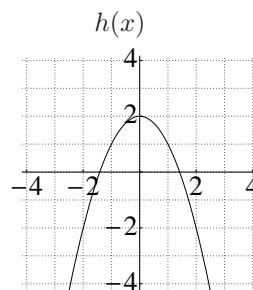
Example 4. In each of the two parts below, you are given a function f and are shown two transformations, g and h , of f . Describe in words how g is obtained from f , and how h is obtained from g , and then find formulas for g and h . Check your answers either using a graphing calculator or by plotting test points.

(a)



Reflect about x -axis

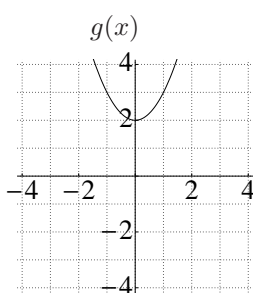
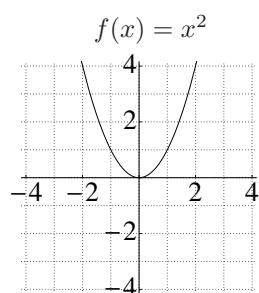
$$\begin{aligned} g(x) &= -f(x) \\ &= -x^2 \end{aligned}$$



Shift up 2 units

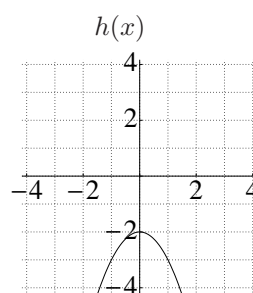
$$\begin{aligned} h(x) &= g(x) + 2 \\ &= -x^2 + 2 \end{aligned}$$

(b)



Shift up 2 units

$$\begin{aligned} g(x) &= f(x) + 2 \\ &= x^2 + 2 \end{aligned}$$



Reflect about x -axis

$$\begin{aligned} h(x) &= -g(x) \\ &= -(x^2 + 2) \\ &= -x^2 - 2 \end{aligned}$$

Note: In parts (a) and (b) above, the same two transformations were applied to f , but in different orders. However, even though the same two transformations were applied in both cases, the resulting functions were not the same.

Examples and Exercises

1. Write formulas for each of the following transformations of the function $q(p) = p^2 - p + 1$.

(a) $q(p - 1)$

(b) $q(p) - 1$

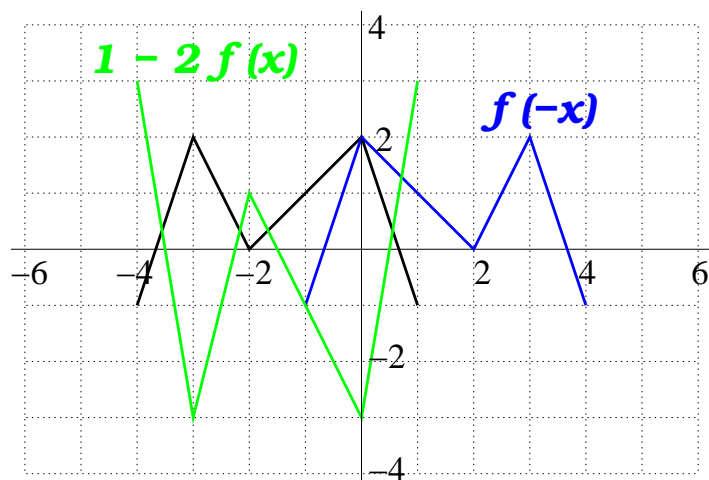
(c) $-2q(-p)$

(a) $q(p - 1) = (p - 1)^2 - (p - 1) + 1 = p^2 - 2p + 1 - p + 1 + 1 = p^2 - 3p + 3.$

(b) $q(p) - 1 = (p^2 - p + 1) - 1 = p^2 - p.$

(c) $-2q(-p) = -2((-p)^2 - (-p) + 1) = -2(p^2 + p + 1) = -2p^2 - 2p - 2.$

2. Let $y = f(x)$ be the function whose graph is given below. Fill in the entries in the table below, and then sketch a graph of the transformations $y = f(-x)$ and $y = 1 - 2f(x)$.



x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$f(x)$	--	--	-1	2	0	1	2	-1	--	--	--	--	--
$f(-x)$	--	--	--	--	--	-1	2	1	0	2	-1	--	--
$1 - 2f(x)$	--	--	3	-3	1	-1	-3	3	--	--	--	--	--

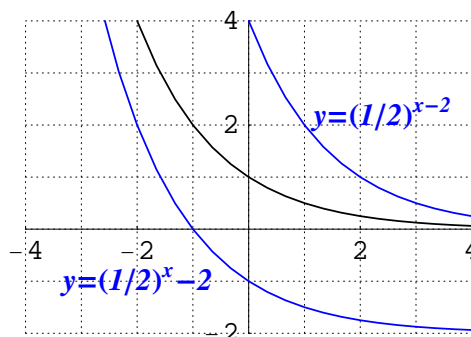
Note that since the domain of $f(x)$ is $-4 \leq x \leq 1$, the domain of $f(-x)$ is $-1 \leq x \leq 4$ because $f(-x)$ is the reflection of $f(x)$ through the y -axis. We calculate several of the output values of $f(-x)$ below:

$$\begin{aligned}
 x = -1 &\implies y = f(-x) = f(-(-1)) = f(1) = -1 \\
 x = 0 &\implies y = f(-x) = f(-0) = f(0) = 2 \\
 x = 1 &\implies y = f(-x) = f(-1) = 1 \\
 &\vdots
 \end{aligned}$$

Similarly, we calculate some values for $1 - 2f(x)$:

$$\begin{aligned}
 x = -4 &\implies y = 1 - 2f(x) = 1 - 2f(-4) = 1 - 2 \cdot (-1) = 3 \\
 x = -3 &\implies y = 1 - 2f(x) = 1 - 2f(-3) = 1 - 2 \cdot 2 = -3 \\
 x = -2 &\implies y = 1 - 2f(x) = 1 - 2f(-2) = 1 - 2 \cdot 0 = 1 \\
 &\vdots
 \end{aligned}$$

3. Given to the right is the graph of the function $y = \left(\frac{1}{2}\right)^x$. On the same set of axes, sketch the graph of $y = \left(\frac{1}{2}\right)^{x-2}$ and $y = \left(\frac{1}{2}\right)^x - 2$.



Let $f(x) = (1/2)^x$. Then

$$\begin{aligned}
 f(x-2) &= \left(\frac{1}{2}\right)^{x-2} \longleftarrow \text{right shift by 2} \\
 f(x) - 2 &= \left(\frac{1}{2}\right)^x - 2 \longleftarrow \text{down shift by 2}
 \end{aligned}$$

4. Let $H = f(t)$ be the temperature of a heated office building t hours after midnight. (See diagram to the right for a graph of f .) Write down a formula for a new function that matches each story below.

- (a) The manager decides that the temperature should be lowered by 5 degrees throughout the day.

Here, we need to lower the output value by 5 degrees for every value of t . Therefore, our answer is

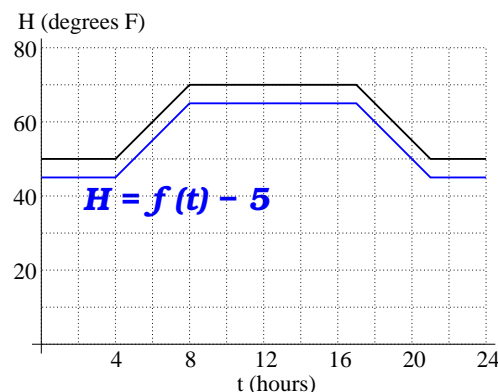
$$H = f(t) - 5,$$

which represents a vertical shift down by 5.

- (b) The manager decides that employees should come to work 2 hours later and leave 2 hours later.

In this case, we want each temperature reading on the graph of $H = f(t)$ above to occur 2 hours later. Graphically, this means we need to shift the graph to the right 2 units. Therefore, our answer is

$$H = f(t - 2).$$

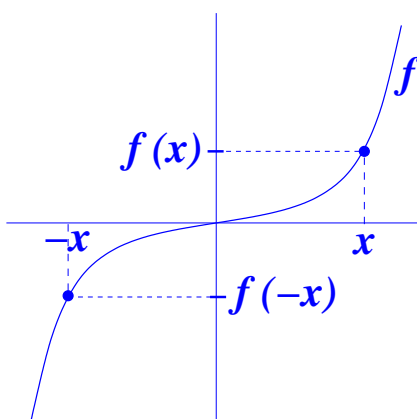
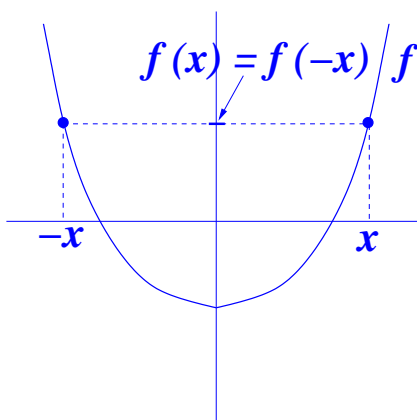


Definition

We say that a function is *even* if $f(-x) = f(x)$ for all x in the domain of the function. In other words, an even function is symmetric about the y -axis.

We say that a function is *odd* if $f(-x) = -f(x)$ for all x in the domain of the function. In other words, an odd function is symmetric about the origin.

Sketch



5. Use algebra to show that $f(x) = x^4 - 2x^2 + 1$ is an even function and that $g(x) = x^3 - 5x$ is an odd function.

We have

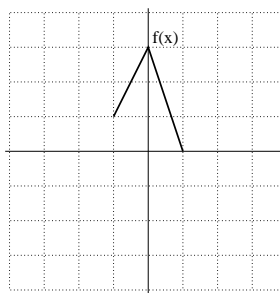
$$\begin{aligned} f(-x) &= (-x)^4 - 2(-x)^2 + 1 \\ &= x^4 + 2x^2 + 1 \\ &= f(x), \end{aligned}$$

so f is even because $f(x) = f(-x)$ for all x . Similarly, we have

$$\begin{aligned} g(-x) &= (-x)^3 - 5(-x) = (-1)^3 x^3 + 5x = -x^3 + 5x \\ &= -(x^3 - 5x) \\ &= -g(x), \end{aligned}$$

so g is odd because $g(-x) = -g(x)$ for all x .

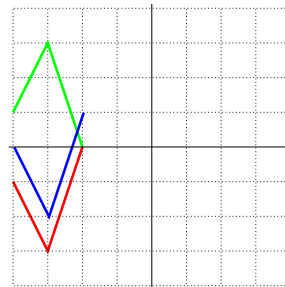
6. Given the graph of $y = f(x)$ to the right, sketch the graph of the following related functions:



(a) $y = -f(x + 3) + 1$

$$\begin{aligned} y &= f(x + 3) && \leftarrow \text{left shift (green graph)} \\ y &= -f(x + 3) && \leftarrow \text{reflect about } x\text{-axis (red graph)} \\ y &= -f(x + 3) + 1 && \leftarrow \text{up shift (blue graph)} \end{aligned}$$

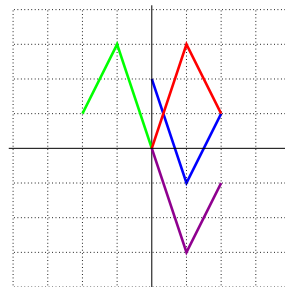
Our final answer is the blue graph to the right.



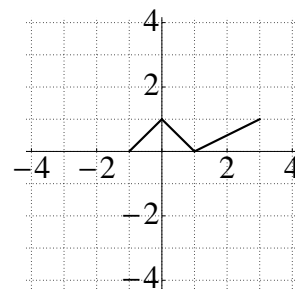
(b) $y = 2 - f(1 - x)$

$$\begin{aligned} y &= f(1 + x) && \leftarrow \text{left shift (green graph)} \\ y &= f(1 + (-x)) && \leftarrow \text{reflect about } y\text{-axis (red graph)} \\ y &= -f(1 - x) && \leftarrow \text{reflect about } x\text{-axis (purple graph)} \\ y &= 2 - f(1 - x) && \leftarrow \text{up shift (blue graph)} \end{aligned}$$

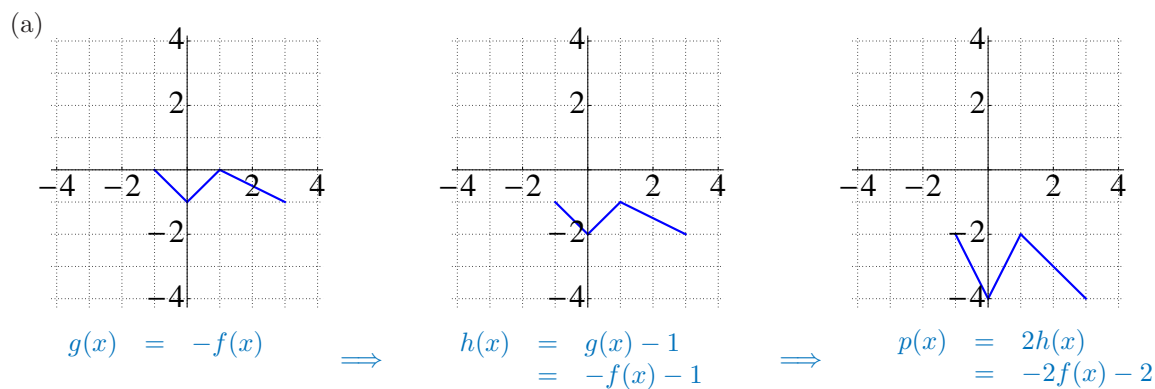
Our final answer is the blue graph to the right.



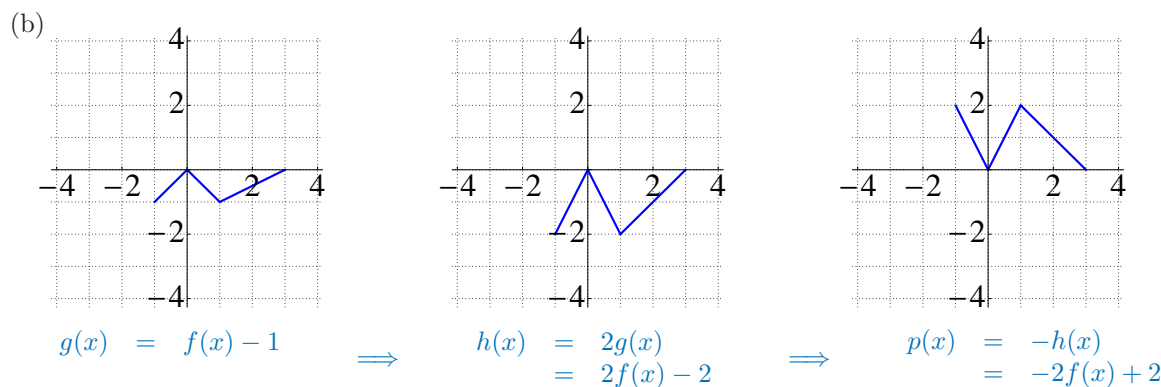
7. To the right, you are given the graph of a function f . Find formulas for the following transformations of f , and also sketch the intermediate transformations.



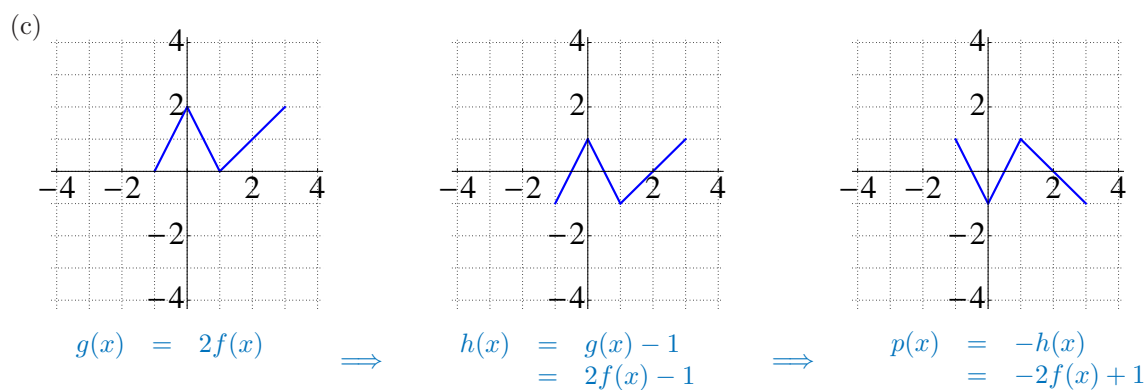
- Reflection about the x -axis, followed by a shift down 1 unit, followed by a vertical stretch by a factor of 2.
- Shift down 1 unit, followed by a vertical stretch by a factor of 2, followed by a reflection about the x -axis.
- Vertical stretch by a factor of 2, followed by a shift down 1 unit, followed by a reflection about the x -axis.



Our answer is therefore $p(x) = -2f(x) - 2$.



Our answer is therefore $p(x) = -2f(x) + 2$.



Our answer is therefore $p(x) = -2f(x) + 1$.