Section 5.1 – Logarithms and Their Properties

Definition. If x is a positive number, then

1. $\log x$ is `the power of 10 that gives` x.
2. $\ln x$ is `the power of e that gives` x.

In other words,

$$y = \log x \quad \text{means that} \quad 10^y = x.$$

$$y = \ln x \quad \text{means that} \quad e^y = x.$$

Example 1. Calculate the following, exactly if possible.

(a) $\log 100 = 2$ (since $10^2 = 100$)

(b) $\log 0.1 = -1$ (since $10^{-1} = 0.1$)

(c) $\ln e = 1$ (since $e^1 = e$)

(d) $\ln(e^2) = 2$

(e) $\log 5 \approx 0.69897$ `(using a calculator)`

Facts about Logarithms.

1. (a) $\log(10^x) = x$ $\qquad \ln(e^x) = x$

   (b) $10^{\log x} = x$ $\qquad e^{\ln x} = x$

2. (a) $\log(ab) = \log a + \log b$ $\qquad \ln(ab) = \ln a + \ln b$

   (b) $\log(\frac{a}{b}) = \log a - \log b$ $\qquad \ln(\frac{a}{b}) = \ln a - \ln b$

   (c) $\log(b^t) = t \log b$ $\qquad \ln(b^t) = t \ln b$

3. (a) $\log 1 = 0$ $\qquad \ln 1 = 0$

   (b) $\log 10 = 1$ $\qquad \ln e = 1$

**Caution!! $\log(ab^t)$ does not equal $t\log(ab)$.

Example 2. Solve each of the following equations for x.

(a) $5 \cdot 4^x = 25$

(b) $5x^4 = 25$

(a) We have

$$5 \cdot 4^x = 25 \implies 4^x = 5 \implies \ln(4^x) = \ln 5 \implies x \ln 4 = \ln 5,$$

so our answer is $x = \dfrac{\ln 5}{\ln 4} \approx 1.161$.

(b) We have

$$5x^4 = 25 \implies x^4 = 5 \implies (x^4)^{1/4} = 25^{1/4} \implies x = \sqrt[4]{25},$$

so our answer is $x = \sqrt[4]{25} \approx 2.236$.

Examples and Exercises

1. Solve each of the following equations for x.

(a) $5 \cdot 3^x = 2 \cdot 7^x$

We have

$$\begin{aligned} & 5 \cdot 3^x = 2 \cdot 7^x \\ \Longrightarrow\quad & \ln(5 \cdot 3^x) = \ln(2 \cdot 7^x) \\ \Longrightarrow\quad & \ln 5 + \ln 3^x = \ln 2 + \ln 7^x \\ \Longrightarrow\quad & \ln 5 + x \ln 3 = \ln 2 + x \ln 7 \\ \Longrightarrow\quad & x \ln 3 - x \ln 7 = \ln 2 - \ln 5 \\ \Longrightarrow\quad & x(\ln 3 - \ln 7) = \ln 2 - \ln 5, \end{aligned}$$

so our answer is $\dfrac{\ln 2 - \ln 5}{\ln 3 - \ln 7}$.

(b) $10e^{4x+1} = 20$

We have

$$\begin{aligned} 10e^{4x+1} = 20 \quad & \Longrightarrow \quad e^{4x+1} = 2 \\ & \Longrightarrow \quad \ln(e^{4x+1}) = \ln 2 \\ & \Longrightarrow \quad 4x + 1 = \ln 2 \\ & \Longrightarrow \quad 4x = \ln 2 - 1, \end{aligned}$$

so our answer is $x = \dfrac{\ln 2 - 1}{4}$.

(c) $a \cdot b^t = c \cdot d^{2t}$

We have

$$\begin{aligned} a \cdot b^t = c \cdot d^{2t} \quad & \Longrightarrow \quad \ln(a \cdot b^t) = \ln(c \cdot d^{2t}) \\ & \Longrightarrow \quad \ln a + \ln(b^t) = \ln c + \ln(d^{2t}) \\ & \Longrightarrow \quad \ln a + t \ln b = \ln c + 2t \ln d \\ & \Longrightarrow \quad t \ln b - 2t \ln d = \ln c - \ln a \\ & \Longrightarrow \quad t(\ln b - 2 \ln d) = \ln c - \ln a, \end{aligned}$$

so our answer is $t = \dfrac{\ln c - \ln a}{\ln b - 2 \ln d}$.

(d) $5x^9 = 10$

We have

$$\begin{aligned} 5x^9 = 10 \quad & \Longrightarrow \quad x^9 = 2 \\ & \Longrightarrow \quad (x^9)^{1/9} = 2^{1/9} \\ & \Longrightarrow \quad x = 2^{1/9}, \end{aligned}$$

so our answer is $x = 2^{1/9}$, or $\sqrt[9]{2}$.

(e) $e^{2x} + e^{2x} = 1$

We have

$$\begin{aligned} e^{2x} + e^{2x} = 1 \quad & \Longrightarrow \quad 2e^{2x} = 1 \\ & \Longrightarrow \quad e^{2x} = \frac{1}{2} \\ & \Longrightarrow \quad \ln(e^{2x}) = \ln\left(\frac{1}{2}\right) \\ & \Longrightarrow \quad 2x = \ln\left(\frac{1}{2}\right), \end{aligned}$$

so our answer is $x = \dfrac{\ln(1/2)}{2}$.

(f) $\ln(x+5) = 10$

We have

$$\begin{aligned} \ln(x+5) = 10 \quad & \Longrightarrow \quad e^{\ln(x+5)} = e^{10} \\ & \Longrightarrow \quad x + 5 = e^{10} \\ & \Longrightarrow \quad x = e^{10} - 5, \end{aligned}$$

so our answer is $x = e^{10} - 5$.

2. Simplify each of the following expressions.

(a) $\log(2A) + \log(B) - \log(AB)$

(b) $\ln(ab^t) - \ln((ab)^t) - \ln a$

(a) We have

$$\log(2A) + \log(B) - \log(AB) = \log(2AB) - \log(AB) = \log\left(\frac{2AB}{AB}\right) = \log 2,$$

so our final answer is $\log 2$.

(b) We have

$$\begin{aligned}\ln(ab^t) - \ln((ab)^t) - \ln a &= \ln a + \ln(b^t) - t\ln(ab) - \ln a = \ln(b^t) - t(\ln a + \ln b)\\ &= t\ln b - t\ln a - t\ln b\\ &= -t\ln a,\end{aligned}$$

so our final answer is $-t\ln a$.

3. Decide whether each of the following statements are true or false.

(a) $\ln(x + y) = \ln x + \ln y$

False. To illustrate this, let $x = 1$ and $y = 1$. Then

$$\begin{aligned}\ln(x+y) &= \ln(1+1) = \ln 2\\ \ln x + \ln y &= \ln 1 + \ln 1 = 0,\end{aligned}$$

so the statement is therefore false because $\ln 2 \neq 0$.

(b) $\ln(x + y) = (\ln x)(\ln y)$

False. To illustrate this, let $x = 1$ and $y = 1$. Then

$$\begin{aligned}\ln(x+y) &= \ln(1+1) = \ln 2\\ (\ln x)(\ln y) &= (\ln 1)(\ln 1) = 0,\end{aligned}$$

so the statement is false because $\ln 2 \neq 0$.

(c) $\ln(ab^2) = \ln a + 2\ln b$

This statement is true because, using properties of logs, we have

$$\ln(ab^2) = \ln a + \ln(b^2) = \ln a + 2\ln b.$$

(d) $\ln(ab^x) = \ln a + x\ln b$

This statement is true because, using properties of logs, we have

$$\ln(ab^x) = \ln a + \ln(b^x) = \ln a + x\ln b.$$

(e) $\ln(1/a) = -\ln a$

This statement is true, because, using properties of logs, we have

$$\ln(1/a) = \ln 1 - \ln a = 0 - \ln a = -\ln a.$$

Section 5.2 – Logarithms and Exponential Models

Review: Two ways of writing exponential functions:

(1) $Q = ab^t$ (2) $Q = ae^{kt}$

$$\begin{aligned} a &= \texttt{initial amount} \\ b &= 1+r, \quad \texttt{where } r = \texttt{growth or decay rate} \\ k &= \texttt{continuous growth or decay rate} \end{aligned}$$

Example 1. Fill in the gaps in the chart below, assuming that t is measured in years:

Formula		**Growth or Decay Rate**	
$Q = ab^t$	$Q = ae^{kt}$	Per Year	Continuous Per Year
$Q = 6(0.9608)^t$	$Q = 6e^{-0.04t}$	-3.92%	-4%
$Q = 5(1.2)^t$	$Q = 5e^{0.1823t}$	20%	18.23%
$Q = 10(0.91)^t$	$Q = 10e^{-0.0943t}$	-9%	-9.43%

(A) We have

$$\begin{aligned} Q &= 6e^{-0.04t} && (k = -0.04) \\ &= 6(e^{-0.04})^t \\ &\approx 6(0.9608)^t && (\text{since } e^{-0.04} \approx 0.9608) \end{aligned}$$

Thus, $1 + r = b = 0.9608$, so $r = 0.9608 - 1 = -0.0392$.

(B) We have

$$\begin{aligned} Q &= 5(1.2)^t && (1 + r = 1.2 \implies r = 0.2) \\ &= 5(e^{\ln 1.2})^t \\ &\approx 5e^{0.1823t}, \end{aligned}$$

so $k = 0.1823$.

(C) We have

$$\begin{aligned} Q &= 10(0.91)^t && (1 + r = 0.91 \implies r = -0.09) \\ &= 10(e^{\ln 0.91})^t \\ &\approx 10e^{-0.0943t}, \end{aligned}$$

so $k = -0.0943$.

Example 2. The population of a bacteria colony starts at 100 and grows by 30% per hour.

(a) Find a formula for the number of bacteria, P, after t hours.

Since P is an exponential function of t, we know that

$$P = ab^t.$$

We are also given that the starting number of bacteria is $a = 30$, and the hourly growth rate is 30%, which means that $r = 0.3$. Therefore,

$$b = 1 + r = 1.3,$$

and our final answer is $P = 100(1.3)^t$.

(b) What is the doubling time for this population; that is, how long does it take the population to double in size?

Here, we are being asked to solve for t when $P = 200$, which is double the starting population. We have

$$\begin{aligned} P = 200 \quad \implies \quad 200 = 100(1.3)^t \quad &\implies \quad 2 = (1.3)^t \\ &\implies \quad \ln 2 = t \ln 1.3 \\ &\implies \quad \frac{\ln 2}{\ln 1.3} = t, \end{aligned}$$

so the doubling time is $t = (\ln 2)/(\ln 1.3) \approx 2.64$ hours.

(c) What is the continuous growth rate of the colony?

We have

$$P = 100(1.3)^t \quad \implies \quad P = 100(\mathrm{e}^{\ln 1.3})^t \quad \implies \quad P \approx 100\mathrm{e}^{0.2624t},$$

so $k = 0.2624$. Therefore, the continuous growth rate of the colony is 26.24%.

Definition. The half-life of a radioactive substance is the amount of time that it takes for half of a given sample to decay.

Example 3. The half-life of a Twinkie is 14 days.

(a) Find a formula for the amount of Twinkie left after t days.

Let Q be the amount of Twinkie left, in grams, and note that since Q is an exponential function of t, we have

$$Q = ab^t.$$

Since the half-life is 14 days, and the initial amount of Twinkie is given by the constant a, we know that $Q = (1/2)a$ when $t = 14$. Substituting these two values into the above equation, we obtain

$$Q = ab^t \quad \Longrightarrow \quad \frac{1}{2}a = ab^{14} \quad \Longrightarrow \quad \frac{1}{2} = b^{14} \quad \text{(after dividing both sides by } a\text{)}$$
$$\Longrightarrow \quad (0.5)^{1/14} = b,$$

so $b = (0.5)^{1/14} \approx 0.9517$. Therefore, our final answer is $Q = a((0.5)^{1/14})^t = a(0.5)^{t/14}$, or $Q \approx a(0.9517)^t$.

(b) Find the daily decay rate of the Twinkie.

Since $Q \approx a(0.9514)^t$ from part (a), we see have

$$1 + r = b = 0.9514 \quad \Longrightarrow \quad r = 0.9514 - 1 = -0.0483,$$

so the daily decay rate is 4.83%.

Examples and Exercises

1. Scientists observing owl and hawk populations collect the following data. Their initial count for the owl population is 245 owls, and the population grows by 3% per year. They initially observe 63 hawks, and this population doubles every 10 years.

   (a) Find formulas for the size of the population of owls and hawks as functions of time.

   Let $P_1 = ab^t$ represent the owl population after t years. Since we are given that the starting population of owls is $a = 245$ and that the annual growth rate is $r = 0.03$, we have $b = 1 + r = 1.03$. Therefore, our formula for the owl population is

   $$P_1 = 245(1.03)^t.$$

   Now, let $P_2 = cd^t$ represent the hawk population after t years. This time, the initial population is $c = 63$, and since the population doubles every 10 years, we know that $P_2 = 126$ when $t = 10$. Therefore, we have

   $$\begin{aligned} P_2 = cd^t \quad \Longrightarrow \quad 126 = 63 \cdot d^{10} \quad &\Longrightarrow \quad 2 = d^{10} \\ &\Longrightarrow \quad 2^{1/10} = d, \end{aligned}$$

   so our formula for the hawk population is

   $$P_2 = 63(2^{1/10})^t = 63 \cdot 2^{t/10}, \qquad \text{or} \qquad P_2 = 63(1.0718)^t.$$

   (b) When will the populations be equal?

   We have

   $$\begin{aligned} P_1 = P_2 \quad &\Longrightarrow \quad 245(1.03)^t = 63 \cdot 2^{t/10} \\ &\Longrightarrow \quad \ln(245(1.03)^t) = \ln(63 \cdot 2^{t/10}) \\ &\Longrightarrow \quad \ln 245 + t \ln 1.03 = \ln 63 + (t/10) \ln 2 \\ &\Longrightarrow \quad t \ln 1.03 - 0.1t \ln 2 = \ln 63 - \ln 245 \\ &\Longrightarrow \quad t(\ln 1.03 - 0.1 \ln 2) = \ln 63 - \ln 245, \end{aligned}$$

   so $t = \dfrac{\ln 63 - \ln 245}{\ln 1.03 - 0.1 \ln 2} \approx 34.2$. Therefore, the populations will be equal after about 34.2 years.

2. Find the half-lives of each of the following substances.

(a) Tritium, which decays at an annual rate of 5.471% per year.

Since we are given that $r = -0.05471$, we have $b = 1 + r = 0.94529$. Therefore, $Q = a(0.94529)^t$ is the amount of Tritium left after t years, where a is the initial amount present. We want to find t when $Q = 0.5a$ (half the starting amount), so we have

$$0.5a = a(0.94529)^t \implies 0.5 = (0.94529)^t \implies \ln 0.5 = t \ln 0.94529,$$

so the half life is $t = (\ln 0.5)/(\ln 0.94529) \approx 12.32$ years.

(b) Vikinium, which decays at a continuous rate of 10% per week.

Since we are given that $k = -0.10$, we have $Q = ae^{-0.1t}$, where Q is the amount of Vikinium left after t weeks, and a is the starting amount. Again, we want to find t when $Q = 0.5a$, so we have

$$0.5a = ae^{-0.1t} \implies 0.5 = e^{-0.1t} \implies \ln 0.5 = -0.1t,$$

so the half life is $t = (\ln 0.5)/(-0.1) \approx 6.93$ weeks.

3. If 17% of a radioactive substance decays in 5 hours, how long will it take until only 10% of a given sample of the substance remains?

Let Q represent the amount of the substance remaining after t hours, so that

$$Q = ab^t.$$

Since 17% of the substance decays in 5 hours, this means that 83% of it will remain after 5 hours; that is, Q is 83% of the starting amount after 5 hours. In other words, we know that when $t = 5$, $Q = 0.83a$. Therefore, we have

$$\begin{aligned} Q = ab^t \quad \implies \quad 0.83a = ab^5 \quad &\implies \quad 0.83 = b^5 \\ &\implies \quad (0.83)^{1/5} = b, \end{aligned}$$

so our formula becomes

$$Q = a((0.83)^{1/5})^t = a(0.83)^{t/5}.$$

We are being asked to find t when Q is 10% of the starting amount; that is, $Q = 0.1a$. Substituting into the above formula, we have

$$\begin{aligned} 0.1a = a(0.83)^{t/5} \quad \implies \quad 0.1 = (0.83)^{t/5} \quad &\implies \quad \ln 0.1 = \ln((0.83)^{t/5}) \\ &\implies \quad \ln 0.1 = (t/5)\ln 0.83 \\ &\implies \quad 5\ln 0.1 = t\ln 0.83, \end{aligned}$$

so $t = (5\ln 0.1)/\ln 0.83 \approx 61.79$. Therefore, it takes about 61.79 hours until only 10% of the sample remains.

Section 5.3 – The Logarithmic Function

1. Consider the functions $f(x) = \ln x$ and $g(x) = \log x$.

   (a) Complete the table below.

x	0.1	0.5	1	2	4	6	8	10
$\ln x$	-2.30	-0.693	0	0.693	1.386	1.792	2.079	2.303
$\log x$	-1	-0.301	0	0.301	0.602	0.778	0.903	1

   (b) Plug a few very small numbers x into $\ln x$ and $\log x$ (like 0.01, 0.001, etc.) What happens to the output values of each function?

$$\begin{aligned} \ln(0.01) &= -4.605 \\ \ln(0.001) &= -6.908 \\ \ln(0.0001) &= -9.21034 \end{aligned} \qquad \begin{aligned} \log(0.01) &= -2 \\ \log(0.001) &= -3 \\ \log(0.0001) &= -4 \end{aligned}$$

   It appears that the output values are becoming more and more negative for both functions. It is clear from the pattern that $\log x$ approaches $-\infty$ as x gets closer and closer to 0. In fact, the same behavior is true for $\ln x$ since $\ln(e^{-1}) = -1,\ \ln(e^{-2}) = -2,\ \ln(e^{-3}) = -3,$ etc.

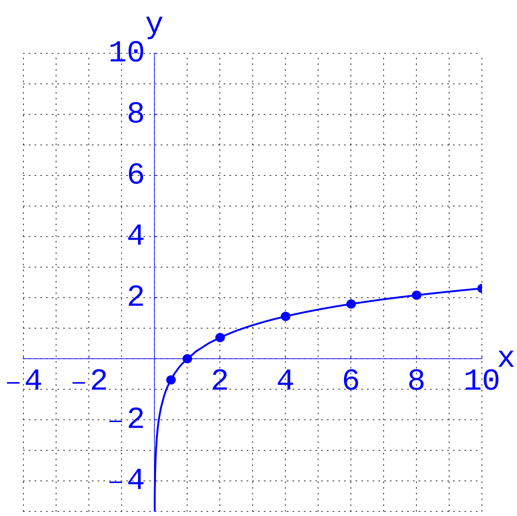
   (c) If you plug in $x = 0$ or negative numbers for x, are $\ln x$ and $\log x$ defined? Explain.

   No, because there is no power of 10 (or of e) that gives 0 or a negative number.

   (d) What is the domain of $f(x) = \ln x$? What is the domain of $g(x) = \log x$?

   The domain of both functions is $\{x : x > 0\}$.

   (e) Sketch a graph of $f(x) = \ln x$ below, choosing a reasonable scale on the x and y axes. Does $f(x)$ have any vertical asymptotes? Any horizontal asymptotes?



   f has a vertical asymptote at $x = 0$ but no horizontal asymptote. Note that it does not have a horizontal asymptote because $\ln x$ approaches ∞ as x approaches ∞: $\ln(e) = 1,\ \ln(e^2) = 2,\ \ln(e^3) = 3,$ etc.

2. What is the domain of the following four functions?

   (a) $y = \ln(x^2)$
   (b) $y = (\ln x)^2$
   (c) $y = \ln(\ln x)$
   (d) $y = \ln(x - 3)$

   (a) In order for $\ln(x^2)$ to be defined, we need $x^2 > 0$. Since any number squared is positive (except 0), the domain is $\{x : x \neq 0\}$; that is, all real numbers except 0.

   (b) Here, the variable x is being substituted into $\ln x$, so we must have $x > 0$. Next, since any number can be squared, there are no further restrictions on x, so the domain is $\{x : x > 0\}$.

   (c) In order for $\ln x$, the inside function, to be defined, we need $x > 0$. On the other hand, in order for $\ln(\ln x)$ to be defined, we need $\ln x > 0$. Therefore, since $\ln x$ is the power of e needed to get x, $\ln x > 0$ means that $x > 1$. Thus, we need $x > 0$ and $x > 1$ to both be true, so the domain is $\{x : x > 1\}$.

   (d) In order for $\ln(x - 3)$ to be defined, we need $x - 3 > 0$, which means that $x > 3$. Therefore, the domain is $\{x : x > 3\}$.

3. Consider the exponential functions $f(x) = e^x$ and $g(x) = e^{-x}$. What are the domains of these two functions? Do they have any horizontal asymptotes? any vertical asymptotes?

   Since e^x and e^{-x} can be calculated for any value of x, the domain of both f and g is all real numbers. Both functions have a horizontal asymptote at $y = 0$ but no vertical asymptote.