Chapter 4 Skills – Exponents

Properties of Exponents

1. $a^n a^m = \underline{a^{n+m}}$

2. $\dfrac{a^n}{a^m} = \underline{a^{n-m}}$

3. $(a^m)^n = \underline{a^{mn}}$

4. $(ab)^n = \underline{a^n b^n}$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{\underline{a^n}}{\underline{b^n}}$

Caution!!

$(a+b)^n$ does <u>not</u>, in general, equal $a^n + b^n$! For example, $(a+b)^2 = a^2 + 2ab + b^2$, which is <u>not</u> the same as $a^2 + b^2$.

Some Definitions

(A) $a^0 = \underline{\quad 1 \quad}$

(B) $a^{-n} = \underline{\dfrac{1}{a^n}}$

(C) $a^{\frac{1}{n}} = \underline{\sqrt[n]{a}}$

(D) $a^{\frac{m}{n}} = \underline{\sqrt[n]{a^m}}$

Example 1. Without a calculator, simplify $9^{-1/2} + \sqrt{0.01}$.

We have

$$
\begin{aligned}
9^{-1/2} + \sqrt{0.01} &= \frac{1}{9^{1/2}} + \left(\frac{1}{100}\right)^{1/2} = \frac{1}{9^{1/2}} + \frac{1^{1/2}}{100^{1/2}} \\
&= \frac{1}{\sqrt{9}} + \frac{\sqrt{1}}{\sqrt{100}} \\
&= \frac{1}{3} + \frac{1}{10} = \frac{10}{30} + \frac{3}{30},
\end{aligned}
$$

so our final answer is $\dfrac{13}{30}$.

Example 2. Simplify $\sqrt{x^e y^{e/2}} + (x^e)(x^e)^2$.

We have

$$
\begin{aligned}
\sqrt{x^e y^{e/2}} + (x^e)(x^e)^2 &= (x^e y^{e/2})^{1/2} + x^e x^{2e} = (x^e)^{1/2}(y^{e/2})^{1/2} + x^{3e} \\
&= x^{e/2} y^{e/4} + x^{3e}.
\end{aligned}
$$

Example 3. Simplify both of the following: (a) $\dfrac{n^{-1}a}{a^2}$ (b) $\dfrac{n^{-1}a+1}{a^2}$

(a) We have

$$\frac{n^{-1}a}{a^2} \;=\; \frac{\frac{1}{n}a}{a^2} \;=\; \frac{\frac{1}{n}}{1}\cdot\frac{a}{a^2} \;=\; \frac{1}{n}\cdot\frac{1}{a} \;=\; \frac{1}{na},$$

so our answer is $\dfrac{1}{na}$.

Shortcut: Since n^{-1}, a, and a^2 are all factors of the given fraction (meaning that only multiplication and division are involved), we can simply move them between the numerator and denominator as long as we change the sign on the exponent:

$$\frac{n^{-1}a}{a^2} \;=\; \frac{a}{n^1a^2} \;=\; \frac{1}{na^2a^{-1}} \;=\; \frac{1}{na}.$$

Note that we obtained the same answer that we did doing it the long way.

(b) We have

$$\frac{n^{-1}a+1}{a^2} \;=\; \frac{\frac{n^{-1}a}{1}+1}{a^2} \;=\; \frac{\frac{a}{n}+1}{a^2} \;=\; \frac{\frac{a}{n}+\frac{n}{n}}{\frac{a^2}{1}}$$

$$= \; \frac{a+n}{n}\cdot\frac{1}{a^2}$$

$$= \; \frac{a+n}{na^2}.$$

Therefore, our final answer is $\dfrac{a+n}{na^2}$.

Examples and Exercises

Directions. For problems 1-7, simplify. For problem 8, solve for x. You may need extra paper for your calculations.

1. $\dfrac{(xy^3)^2}{x^0 y^5}$

$$\frac{(xy^3)^2}{x^0 y^5} = \frac{x^2(y^3)^2}{1 \cdot y^5}$$
$$= \frac{x^2 y^6}{y^5}$$
$$= x^2 y$$

2. $\dfrac{(AB)^4}{A^{-1}B^{-2}}$

$$\frac{(AB)^4}{A^{-1}B^{-2}} = \frac{A^4 B^4}{A^{-1}B^{-2}}$$
$$= A^{4+1}B^{4+2}$$
$$= A^5 B^6$$

3. $\dfrac{a^3 b^{-1}}{\sqrt{a^{5/2}}}$

$$\frac{a^3 b^{-1}}{\sqrt{a^{5/2}}} = \frac{a^3 b^{-1}}{(a^{5/2})^{1/2}}$$
$$= \frac{a^3}{b a^{5/4}}$$
$$= \frac{a^{12/4}}{b a^{5/4}}$$
$$= \frac{a^{7/4}}{b}$$

4. $2b^{-1}(b^2 + b) - 2$

$$2b^{-1}(b^2 + b) - 2 = 2b^{-1+2} + 2b^{-1+1} - 2$$
$$= 2b^1 + 2b^0 - 2$$
$$= 2b + 2 - 2$$
$$= 2b$$

5. $\dfrac{2M + M^{-1}}{1 + 2M^{-2}}$

$$\frac{2M + M^{-1}}{1 + 2M^{-2}} = \frac{2M + \frac{1}{M}}{1 + \frac{2}{M^2}} = \frac{\frac{2M^2}{M} + \frac{1}{M}}{\frac{M^2}{M^2} + \frac{2}{M^2}}$$
$$= \frac{\frac{2M^2 + 1}{M}}{\frac{M^2 + 2}{M^2}}$$
$$= \frac{2M^2 + 1}{M} \cdot \frac{M^2}{M^2 + 2}$$
$$= \frac{M^2(2M^2 + 1)}{M(M^2 + 2)}$$
$$= \frac{2M^3 + M}{M^2 + 2}$$

6. $3\sqrt[3]{t^3 + 7(t^9)^{1/3}}$

$$3\sqrt[3]{t^3 + 7(t^9)^{1/3}} = 3\sqrt[3]{t^3 + 7t^3} = 3\sqrt[3]{8t^3}$$
$$= 3(8)^{1/3}(t^3)^{1/3}$$
$$= 3 \cdot 2 \cdot t$$
$$= 6t$$

7. $\dfrac{2km^3 + k^2 m}{km^{-1}}$

$$\frac{2km^3 + k^2 m}{km^{-1}} = \frac{km(2m^2 + k)}{km^{-1}} = m^{1-(-1)}(2m^2 + k)$$
$$= m^2(2m^2 + k)$$
$$= 2m^4 + km^2$$

8. $81^x = 3$

$$81^x = 3 \implies (3^4)^x = 3 \implies 3^{4x} = 3^1$$
$$\implies 4x = 1$$
$$\implies x = \frac{1}{4}$$

Sections 4.1-4.3 — Exponential Functions

Example 1. The population of a rapidly-growing country starts at 5 million and increases by 10% each year. Complete the table below:

t (years)	P, population (in millions)	ΔP, increase in population (mil)
0	5	
1	5.5	0.5
2	6.05	0.55
3	6.655	0.605
4	7.3205	0.6655

$$
\begin{aligned}
\text{Year 1: } P &= 5 + 5 \cdot (0.1) = 5.5 \quad \longleftarrow 5(1.1) \\
\text{Year 2: } P &= 5.5 + 5.5(0.1) \\
&= 5(1.1) + 5(1.1)(0.1) = 6.05 \quad \longleftarrow 5(1.1)^2 \\
\text{Year 3: } P &= 6.05 + 6.05(0.1) \\
&= 5(1.1)^2 + 5(1.1)^2(0.1) = 6.655 \quad \longleftarrow 5(1.1)^3
\end{aligned}
$$

Definition. An exponential function $Q = f(t)$ has the formula $f(t) = ab^t, \ b > 0,$

where

$$
\begin{aligned}
a &= \text{initial value of } Q \\
b &= \text{growth factor } (b > 1 \text{ gives growth, } 0 < b < 1 \text{ gives decay})
\end{aligned}
$$

Note: $b = 1 + r$, where r is the decimal representation of the percent rate of change.

Example 2.

Description	Growth Factor and Formula
The population, P, of ants in your kitchen starts at 10 and increases by 5% per day.	$r = 0.05$, so $b = 1 + r = 1.05$. Therefore, $P = 10(1.05)^t$.
The value, V, of a 1982 Chevy Caprice starts at \$10000 and decreases by 8% per year.	$r = -0.08$, so $b = 1 + r = 0.92$. Therefore, $V = 10000(0.92)^t$.
The air pressure, A, starts at __960__ millibars at sea level ($h = 0$) and decreases by __20__% per mile increase in elevation.	$A = 960(0.8)^h$

Example 3. Analyze the functions f and g below. Which is linear? Which is exponential? Give a formula for each function.

x	5	10	15	20	25
$f(x)$	10	17	24	31	38
$g(x)$	100	115	132.25	152.09	174.9

First, for the function f, we calculate the change in $f(x)$ over each interval (see table below). Note that $f(x)$ changes by 7 for each change of 5 in the value of x; that is, the difference between successive output values is constant. Therefore, f is a linear function.

x	5	10	15	20	25
$f(x)$	10	17	24	31	38
$\Delta f(x)$		7	7	7	7

Now, we calculate the change in $g(x)$ over each interval (see table below). Note that the change in $g(x)$ varies over each interval of length 5, and so g is <u>not</u> a linear function.

x	5	10	15	20	25
$g(x)$	100	115	132.25	152.09	174.9
$\Delta g(x)$		15	17.25	19.84	22.81

However, note that <u>ratios</u> of successive output values are constant for g:

$$\frac{115}{100} = 1.15, \quad \frac{132.25}{115} = 1.15, \quad \frac{152.09}{132.25} = 1.15, \ldots$$

Since this behavior is consistent with the behavior of the exponential function from Example 1, we will attempt to find a formula for an exponential function that fits g.

<u>Given</u>: $g(x) = ab^x, \ g(5) = 100, \ g(10) = 115$

Using the given information above, we have

$$\begin{array}{ccc} g(5) & = & 100 \\ g(10) & = & 115 \end{array} \implies \begin{array}{ccc} ab^5 & = & 100 \\ ab^{10} & = & 115 \end{array} \implies \frac{115}{100} = \frac{ab^{10}}{ab^5}$$

$$\implies 1.15 = b^5$$
$$\implies b = (1.15)^{1/5} \approx 1.028.$$

Therefore, we have

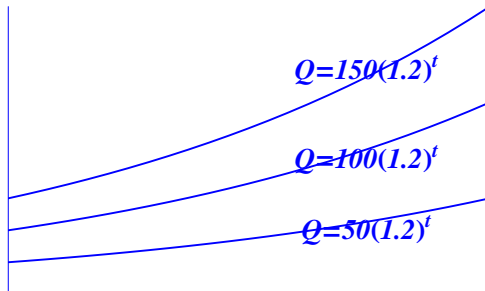$$100 = ab^5 \implies a = \frac{100}{b^5} = \frac{100}{1.15} \approx 86.96,$$

and our final answer is $g(x) \approx 86.96((1.15)^{1/5})^x = 86.96(1.15)^{x/5}$, which can also be written as $g(x) \approx 86.96(1.028)^x$.

Example 4.

Below are the graphs of $Q = 150(1.2)^t$, $Q = 50(1.2)^t$, and $Q = 100(1.2)^t$. Match each formula to the correct graph.

Below are the graphs of $Q = 50(1.2)^t$, $Q = 50(0.6)^t$, $Q = 50(0.8)^t$, and $Q = 50(1.4)^t$. Match each formula to the correct graph.

$Q=150(1.2)^t$

$Q=100(1.2)^t$

$Q=50(1.2)^t$

$Q=50(1.4)^t$ $Q=50(1.2)^t$

$Q=50(0.8)^t$

$Q=50(0.6)^t$
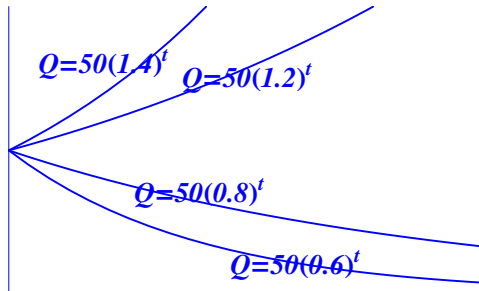
Observations about the graph of $Q = ab^t$:

1. a gives the y-intercept of the graph.

2. If $b > 1$, the function is increasing. If $0 < b < 1$, the function is decreasing.

3. Exponential functions are always concave up.

Examples and Exercises

1. Suppose we start with 100 grams of a radioactive substance that decays by 20% per year. First, complete the table below. Then, find a formula for the amount of the substance as a function of t and sketch a graph of the function.

t (years)	0	1	2	3	4
Q (grams)	100	80	64	51.2	40.96

Given: $Q = ab^t$

We are also given that the initial amount of the substance is $a = 100$ grams, and that the annual decay rate is 20%, so $r = -0.2$. Therefore, $b = 1 + r = 0.8$, and our final answer is $Q = 100(0.8)^t$.
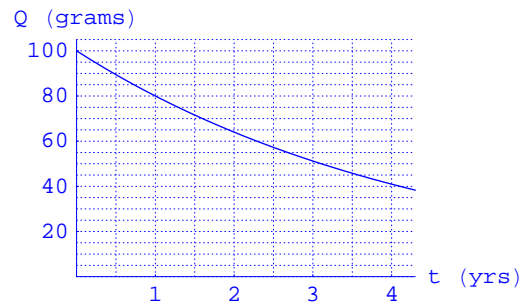
2. Suppose you invest \$10000 in the year 2000 and that the investment earns 4.5% interest annually.

   (a) Find a formula for the value of your investment, V, as a function of time.

   We are given an initial investment of $a = \$10000$, and an annual growth rate of $r = 0.045$. Therefore, $b = 1 + r = 1.045$, and our final answer is $V = 10000(1.045)^t$, where t is time in years since the investment was made.

   (b) What will the investment be worth in 2010? in 2020? in 2030?

   We have

$$f(10) = 10000(1.045)^{10} \approx \$15529.69$$
$$f(20) = 10000(1.045)^{20} \approx \$24117.14$$
$$f(30) = 10000(1.045)^{30} \approx \$37453.18$$

   Therefore, the investment will be worth \$15529.69 in 2010, \$24117.14 in 2020, and \$37453.18 in 2030.

3. The populations of the planet Vulcan and the planet Romulus are recorded in 1980 and in 1990 according to the table below. Also, assume that the population of Vulcan is growing exponentially and that the population of Romulus is growing linearly.

Planet	1980 Population (billions)	1990 Population (billions)
Vulcan	8	12
Romulus	16	20

(a) Find two formulas; one for the population of Vulcan as a function of time and one for the population of Romulus as a function of time. Let $t = 0$ denote the year 1980.

First, we find a population formula for Vulcan. Let P_V be the population of Vulcan, in billions. Then $a = 8$ is the initial population of Vulcan, and we are given that $P_V = 12$ when $t = 10$; therefore, we have

$$P_V = 8 \cdot b^t \quad \implies \quad 12 = 8 \cdot b^{10} \quad \implies \quad \frac{12}{8} = b^{10} \quad \implies \quad \left(\frac{3}{2}\right)^{1/10} = b,$$

so $b = \sqrt[10]{3/2} \approx 1.0414$, and our answer is $P_V = 8(1.5)^{t/10} \approx 8(1.0414)^t$.

Now, we find population formula for Romulus. Since the function for this population is linear, we begin by finding the slope:

$$\frac{\Delta P}{\Delta t} = \frac{20 - 16}{10 - 0} = \frac{2}{5}$$

The table indicates that when $t = 0$, we have $P_R = 16$, so the vertical intercept is 16. Therefore, our final answer is $P_R = \frac{2}{5}t + 16$.

(b) Use your formulas to predict the population of both planets in the year 2000.

$$P_V = 8(1.5)^{20/10} \approx 18 \text{ and } P_R = \frac{2}{5}(20) + 16 = 24,$$

so the population of Vulcan will be about 18 billion and the population of Romulus will be about 24 billion.

(c) According to your formula, in what year will the population of Vulcan reach 50 billion? Explain how you got your answer.

Referring to the graph of P_V as a function of t to the right, it appears that $P_V = 50$ when $t \approx 45$; therefore, Vulcan will reach a population of 50 billion in the year 2025.
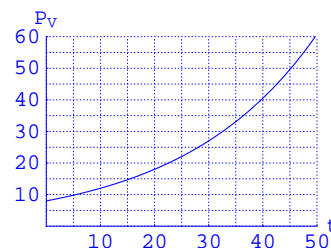
(d) In what year does the population of Vulcan overtake the population of Romulus? Justify your answer with an accurate graph and an explanation.

Referring to the graph, it appears that $P_V = P_R$ at about $t = 32$; therefore, Vulcan overtakes Romulus in total population in about the year 2012.
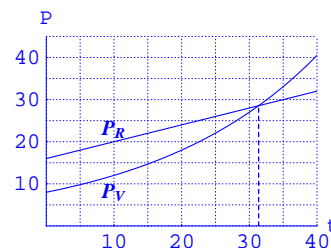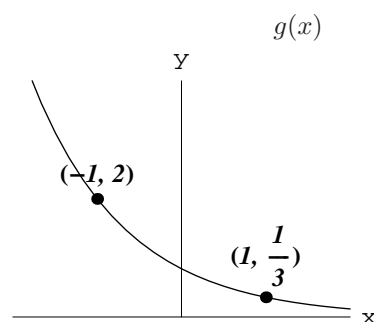
4. Find possible formulas for each of the two functions f and g described below.

x	0	2	4	6
$f(x)$	2	2.5	3.125	3.90625

$g(x)$

For the function f, we know that $f(x) = ab^x$, and we see from the table that $f(0) = 2$, so $a = 2$. Therefore, we have

$$f(2) = 2.5 \implies 2b^2 = 2.5 \implies b^2 = 1.25$$
$$\implies b = \sqrt{1.25}.$$

Therefore, our answer is $f(x) = 2(\sqrt{1.25})^x$, or $f(x) = 2(1.25)^{x/2}$.

Assuming that g is an exponential function, we also know that $g(x) = cd^x$, and from the graph we see that

$$\begin{array}{ll} f(1) &= \frac{1}{3} \\ f(-1) &= 2 \end{array} \implies \begin{array}{ll} ab^1 &= \frac{1}{3} \\ ab^{-1} &= 2 \end{array} \implies \frac{ab}{ab^{-1}} = \frac{1/3}{2} \implies b^2 = \frac{1}{6},$$

which means that $b = \sqrt{1/6}$. Therefore, we have

$$ab^{-1} = 2 \implies a = 2b \implies a = 2\sqrt{1/6},$$

and so our final answer is $g(x) = 2\sqrt{\frac{1}{6}}\left(\sqrt{\frac{1}{6}}\right)^x$, or $g(x) = 2\left(\frac{1}{6}\right)^{(x+1)/2}$.

5. Consider the exponential graphs pictured below and the six constants $a, b, c, d, p,$ and q.

(a) Which of these constants are **definitely** positive?

Since the y-intercept of each graph is above the x-axis, we know that $a, c,$ and p are definitely positive. Also, since the base of an exponential function is always positive, we know that b, d and q are definitely positive. Therefore, all six constants are definitely positive.
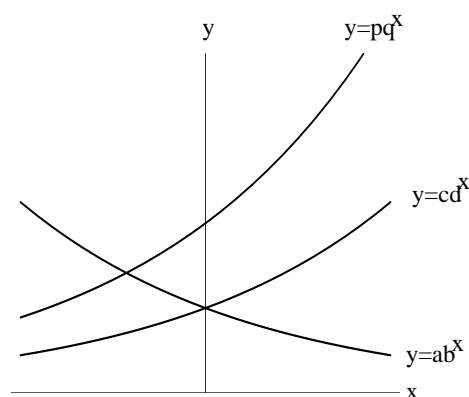
(b) Which of these constants are **definitely** between 0 and 1?

First, notice that q and d are definitely greater than 1 since $y = pq^x$ and $y = cd^x$ are both increasing. Also, since we are not given a scale on the y-axis, we cannot determine whether or not $p, c,$ and a are between 0 and 1 or greater than 1. Finally, because the graph of $y = ab^x$ is decreasing, b is definitely between 0 and 1. Therefore, we conclude that b is the only constant that is definitely between 0 and 1.

(c) Which two of these constants are **definitely** equal?

a and c are definitely equal because $y = cd^x$ and $y = ab^x$ have the same y-intercept.

(d) Which one of the following pairs of constants **could be** equal?

a and p b and d b and q d and q

a and p cannot be equal because the y-intercepts of $y = ab^x$ and $y = pq^x$ are not the same. b and d cannot be equal because $y = cd^x$ is increasing and $y = ab^x$ is decreasing. Similarly, b and q cannot be equal. Therefore, d and q are the only constants from the above list that could be equal.

Section 4.5 – Continuous Growth and the Number e

Preliminary Example. At the In-Your-Dreams Bank of America, all investments earn 100% interest annually. Suppose that you invest $1000 at a time that we will call month 0. Fill in the blanks below to compare what your investment will be worth 1 year later using various methods of interest compounding.

Month	Compounded 1 Time	Compounded 2 Times	Compounded 4 Times
0	$1000	$1000	$1000
1			
2			
3			$1250
4			
5			
6		$1500	$1562.50
7			
8			
9			$1953.13
10			
11			
12	$2000	$2250	$2441.41

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$
$$1000(1+1)^1 \qquad 1000(1+\tfrac{1}{2})^2 \qquad 1000(1+\tfrac{1}{4})^4$$

- After 1 year with n compoundings, we will have $1000(1+\frac{1}{n})^n$ dollars in our account.

- *Continuous compounding* means that we let n approach infinity in the above process. When we do this, we obtain $\$1000e \approx \$2718.28$ as our final balance after twelve months, where $e = 2.7182818\ldots$.

Alternative Formula for Exponential Functions. Given an exponential function $Q = ab^t$, it is possible to rewrite Q as follows:

$$Q = \underline{\ \ a e^{kt}\ \ }$$

The constant k is then called the *continuous growth rate* of Q.

Notes:
- If $k > 0$, then Q is increasing.
- If $k < 0$, then Q is decreasing.

Developed by Jerry Morris

Exercise Suppose that the population of a town starts at 5000 and grows at a continuous rate of 2% per year.

(a) Write a formula for the population of the town as a function of time, in years, after the starting point.

We are given a starting population of $a = 5000$ and a continuous growth rate of $k = 0.02$. Therefore, our final answer is

$$P = 5000e^{0.02t}.$$

(b) What will the population of the town be after 10 years?

Letting $t = 10$, we have

$$t = 10 \quad \implies \quad P = 5000e^{0.02(10)} = 5000e^{0.2} \approx 6107,$$

so the population is about 6107 people.

(c) By what percentage does the population of the town grow each year?

For this question, we wish to find the value of r, the annual growth rate of the population. We can do this by converting the formula for P from part (a) into the form $P = ab^t$. We have

$$\begin{aligned} P = 5000e^{0.02t} &= 5000(e^{0.02})^t \quad \text{(by properties of exponents)} \\ &\approx 5000(1.0202)^t, \end{aligned}$$

so $1 + r = b \approx 1.0202$, meaning that $r \approx 0.0202$. Therefore, we conclude that the town grows by about 2.02% per year.