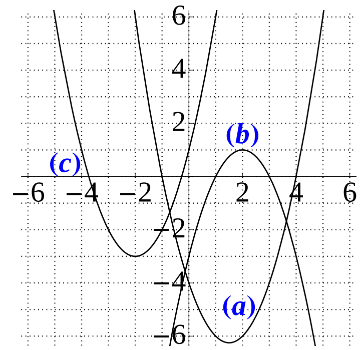
Sections 3.1 & 3.2 – Quadratic Functions



Preliminary Example. To the right, you are given the graphs of three quadratic functions. Without using a graphing calculator, try to match the correct formula below to its graph.

(a) $y = (x+1)(x-4)$ $\longleftarrow$ $y = 0$ when $x = -1$ or $x = 4$

(b) $y = -x^2 + 4x - 3$ $\longleftarrow$ $y = -3$ when $x = 0$

(c) $y = (x+2)^2 - 3$ $\longleftarrow$ $y = -3$ when $x = -2$

In general, a *quadratic function* f can be written in several different ways:

1. $f(x) = ax^2 + bx + c$ (standard form, where a, b, and c are constant)
2. $f(x) = a(x-r)(x-s)$ (factored form, where a, r, and s are constant)
3. $f(x) = a(x-h)^2 + k$ (vertex form, where a, h, and k are constant)

Notes.

- The graph of a quadratic function is called a parabola.
- In factored form, the numbers r and s represent the zeros of f.
- In vertex form, the point (h, k) is called the vertex of the parabola. The axis of symmetry is the line $x = h$. The graph opens upward if $a > 0$ and downward if $a < 0$.

Example 1. Find the vertex of $y = x^2 - 8x - 27$ by completing the square.

$$\begin{aligned} y &= x^2 - 8x - 27 \\ &= (x^2 - 8x \qquad) - 27 \\ &= (x^2 - 8x + 16) - 27 - 16 \qquad \longleftarrow \texttt{since} \left(\frac{-8}{2}\right)^2 = (-4)^2 = 16 \\ &= (x-4)^2 + (-43). \end{aligned}$$

Therefore, $h = 4$ and $k = -43$, and we see that the vertex is $(4, -43)$.

Example 2. Find the vertex of $y = 2x^2 + 7x + 3$ by completing the square.

$$\begin{aligned} y &= 2x^2 + 7x + 3 \\ &= 2\left(x^2 + \frac{7}{2}x \qquad\right) + 3 \\ &= 2\left(x^2 + \frac{7}{2}x + \frac{49}{16}\right) + 3 - 2 \cdot \frac{49}{16} \qquad \longleftarrow \texttt{since} \left(\frac{7/2}{2}\right)^2 = \left(\frac{7}{4}\right)^2 = \frac{49}{16} \\ &= 2\left(x + \frac{7}{4}\right)^2 + \frac{48}{16} - \frac{98}{16} \\ &= 2\left(x - \left(-\frac{7}{4}\right)\right)^2 - \frac{25}{8}, \end{aligned}$$

so the vertex is $\left(-\frac{7}{4}, -\frac{25}{8}\right)$.

Example 3. The height of a rock thrown into the air is given by $h(t) = 32t - 16t^2$ feet, where t is measured in seconds.

(a) Calculate $h(1)$ and give a practical interpretation of your answer.

$h(1) \;=\; 32 - 16(1)^2 \;=\; 32 - 16 \;=\; 16$ feet is the height of the rock after 1 second.

(b) Calculate the zeros of $h(t)$ and explain their meaning in the context of this problem.

Since

$$h(t) = 0 \quad \Longrightarrow \quad 32t - 16t^2 = 0 \quad \Longrightarrow \quad 16t(2-t) = 0 \quad \Longrightarrow \quad 16t = 0 \;\text{ or }\; 2 - t = 0,$$

we see that the zeros are $t = 0$ or $t = 2$. This means that the stone is at ground level at $t = 0$ and $t = 2$ seconds.

(c) Solve the equation $h(t) = 10$ and explain the meaning of your solutions in the context of this problem.

We have

$$32t - 16t^2 = 10 \quad \Longrightarrow \quad 16t^2 - 32t + 10 = 0 \quad \Longrightarrow \quad 8t^2 - 16t + 5 = 0.$$

Using the quadratic formula, the solutions to this equation are given by

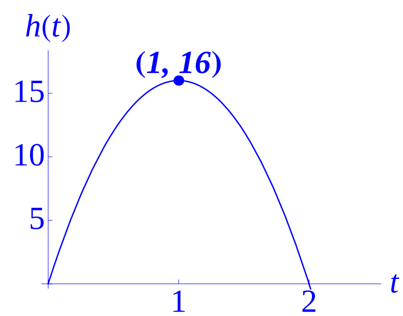
$$t \;=\; \frac{16 \pm \sqrt{(-16)^2 - 4(8)(5)}}{2(8)} \;=\; \frac{16 \pm \sqrt{256 - 160}}{16} \;=\; \frac{16 \pm 4\sqrt{6}}{16} \;=\; \frac{4 \pm \sqrt{6}}{4}.$$

The solutions are therefore $t = \dfrac{4+\sqrt{6}}{4} \approx 1.61$ and $t = \dfrac{4-\sqrt{6}}{4} \approx 0.39$. This means that at $t \approx 1.61$ seconds and $t \approx 0.39$ seconds, the rock is 10 feet above the ground.

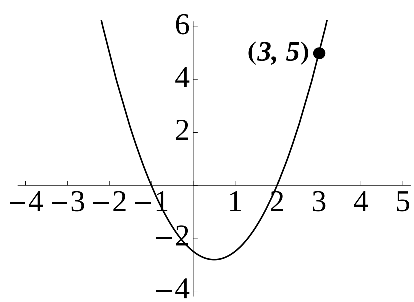
(d) What is the maximum height reached by the stone, and at what time does the maximum height occur?

$$\begin{aligned} -16t^2 + 32t &= -16(t^2 - 2t) \\ &= -16(t^2 - 2t + 1) \;+\; 16 \\ &= -16(t-1)^2 \;+\; 16 \end{aligned}$$

We see from the above calculations that the vertex of this quadratic function is $(1, 16)$, which means that the stone reaches a maximum height of 16 feet at $t = 1$ second.



Example 4. Find a formula for the parabola shown to the right.



Since we can see the zeros of the function from the provided graph, we choose to begin with the factored form of a quadratic function:

<u>Know</u>: $f(x) = a(x+1)(x-2)$

<u>Given</u>: $f(3) = 5$

Since

$$f(3) = 5 \quad \Longrightarrow \quad a(3+1)(3-2) = 5 \quad \Longrightarrow \quad 4a = 5 \quad \Longrightarrow \quad a = \frac{5}{4},$$

we see that a formula for this parabola is $f(x) = \dfrac{5}{4}(x+1)(x-2)$.

Examples and Exercises

1. Find (if possible), the zeros of the following quadratic functions.

   (a) $f(x) = x^2 + 5x - 14$

   We have

   $$x^2 + 5x - 14 = 0 \qquad \Longrightarrow \qquad (x+7)(x-2) = 0,$$

   so either $x + 7 = 0$ or $x - 2 = 0$. Therefore, our final answers are $x = -7$ and $x = 2$.

   (b) $g(x) = x^2 + 1$

   Using the quadratic formula, we have $a = 1, b = 0,$ and $c = 1,$ so that solutions of this equation would look like

   $$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} = \frac{\sqrt{-4}}{2}.$$

   However, since $\sqrt{-4}$ is not a real number, this function has no zeros.

   Alternate Solution: Since $x^2 \geq 0$ is always true, we know that $x^2 + 1 \geq 1$ is always true. Therefore, $g(x)$ can never equal zero.

2. For each of the following, complete the square in order to find the vertex.

   (a) $y = x^2 - 40x + 1$

   $$\begin{aligned} y &= x^2 - 40x + 1 \\ &= (x^2 - 40x \qquad) + 1 \\ &= (x^2 - 40x + 400) + 1 - 400 \qquad \longleftarrow \text{ since } \left(\frac{-40}{2}\right)^2 = (-20)^2 = 400 \\ &= (x-20)^2 - 399 \\ &= (x-20)^2 + (-399), \end{aligned}$$

   so the vertex is $(20, -399)$.

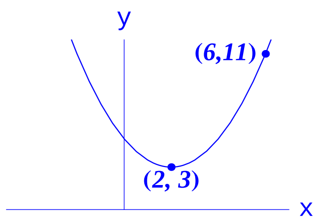
   (b) $y = 2x^2 + 12x + 3$

   $$\begin{aligned} y &= 2x^2 + 12x + 3 \\ &= 2(x^2 + 6x \qquad) + 3 \\ &= 2(x^2 + 6x + 9) + 3 - 2 \cdot 9 \qquad \longleftarrow \text{ since } \left(\frac{6}{2}\right)^2 = 3^2 = 9 \\ &= 2(x+3)^2 - 15, \end{aligned}$$

   so the vertex is $(-3, -15)$.

3. A parabola has its vertex at the point $(2, 3)$ and goes through the point $(6, 11)$. Find a formula for the parabola.

   Given: $y = a(x-2)^2 + 3$
   $f(6) = 11$

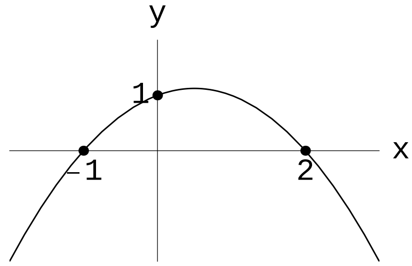
   

   We have

   $$\begin{aligned} f(6) = 11 \quad &\Longrightarrow \quad a(6-2)^2 + 3 = 11 \\ &\Longrightarrow \quad 16a + 3 = 11 \\ &\Longrightarrow \quad 16a = 8, \end{aligned}$$

   so $a = \frac{1}{2}$. Therefore, our final answer is $f(x) = \frac{1}{2}(x-2)^2 + 3$.

4. Find a formula for the quadratic function shown below. Also find the vertex of the function.



<u>Given</u>: $f(x) = a(x+1)(x-2)$

$f(0) = 1$

We have

$$f(0)=1 \implies a(0+1)(0-2)=1 \implies -2a=1 \implies a=-\frac{1}{2}$$

Therefore, a formula for the function is $f(x) = -\frac{1}{2}(x+1)(x-2)$, which is one of our two final answers.

To find the vertex, we must complete the square. We have

$$\begin{aligned} f(x) = -\frac{1}{2}(x^2-x-2) &= -\frac{1}{2}x^2+\frac{1}{2}x+1 \\ &= -\frac{1}{2}(x^2-x \qquad)+1 \\ &= -\frac{1}{2}\left(x^2-x+\frac{1}{4}\right)+1+\frac{1}{2}\cdot\frac{1}{4} \qquad \longleftarrow \text{since } \left(-\frac{1}{2}\right)^2=\frac{1}{4} \\ &= -\frac{1}{2}\left(x-\frac{1}{2}\right)^2+\frac{9}{8}, \end{aligned}$$

so the vertex of the parabola is $\left(\frac{1}{2}, \frac{9}{8}\right)$.

5. A tomato is thrown vertically into the air at time $t=0$. Its height, $d(t)$ (in feet), above the ground at time t (in seconds) is given by $d(t) = -16t^2+48t$.

(a) Find t when $d(t)=0$. What is happening to the tomato the first time that $d(t)=0$? The second time?

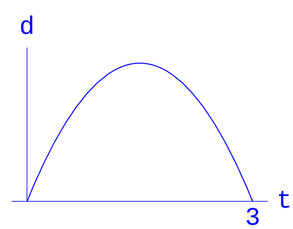
We have

$$\begin{aligned} d(t)=0 \implies -16t^2+48t=0 &\implies 16t(-t+3)=0 \\ &\implies 16t=0 \text{ or } -t+3=0 \\ &\implies t=0 \text{ or } t=3. \end{aligned}$$

Therefore, $t=0$ is the first time at which $d(t)=0$, and this is the time when the tomato is first thrown into the air. Also, $t=3$ is the second time at which $d(t)=0$, and this is when the tomato returns to the ground.

(b) When does the tomato reach its maximum height? How high is the tomato's maximum height?

From our answer to part (a), we know that the graph of $d(t)$ looks roughly like the graph to the right. The maximum height will occur at the vertex, which we calculate below:



$$\begin{aligned} d(t) &= -16(t^2-3t \qquad) \\ &= -16\left(t^2-3t+\frac{9}{4}\right)+16\cdot\frac{9}{4} \qquad \longleftarrow \text{since } \left(-\frac{3}{2}\right)^2=\frac{9}{4} \\ &= -16\left(t-\frac{3}{2}\right)^2+36 \end{aligned}$$

Therefore, the vertex is $(\frac{3}{2}, 36)$, which means that the tomato reaches a maximum height of 36 feet after $\frac{3}{2}=1.5$ seconds.