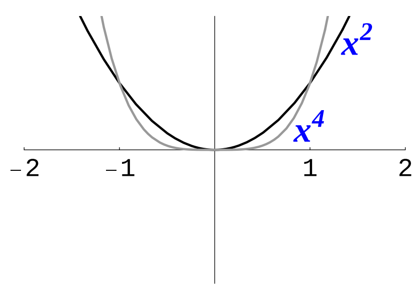
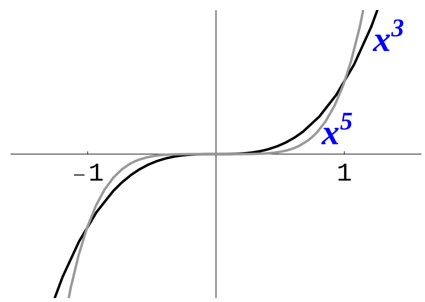
Section 11.1 – Power Functions and Proportionality

Definition. A *power function* is a function of the form $f(x) = kx^p$, where k and p are constants.

I. Positive Integer Powers. Match the following functions to the appropriate graphs below:
$y = x^2, y = x^3, y = x^4, y = x^5$





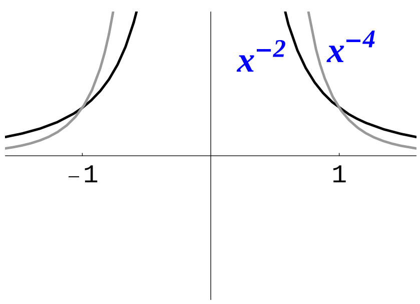
(Even Powers)

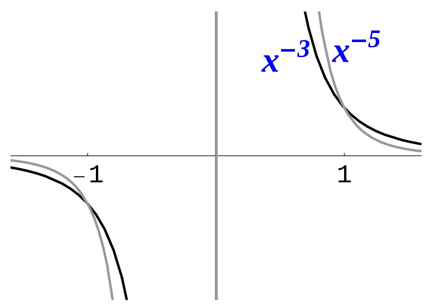
Increasing for $0 < x < \infty$
Decreasing for $-\infty < x < 0$
Concave up everywhere

(Odd Powers)

Increasing everywhere
Concave down for $-\infty < x < 0$
Concave up for $0 < x < \infty$

II. Negative Integer Powers. Match the following functions to the appropriate graphs below:
$y = x^{-2}, y = x^{-3}, y = x^{-4}, y = x^{-5}$





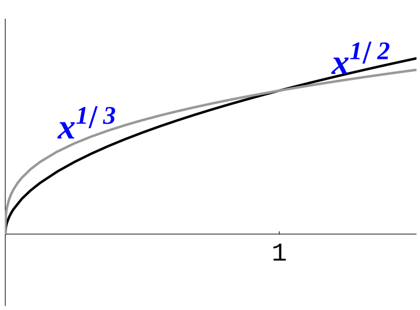
(Even Powers)

Increasing for $-\infty < x < 0$
Decreasing for $0 < x < \infty$
Concave up everywhere

(Odd Powers)

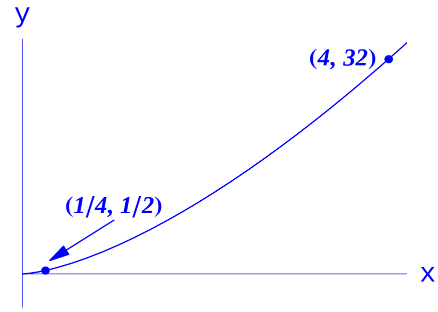
Decreasing everywhere
Concave down for $-\infty < x < 0$
Concave up for $0 < x < \infty$

III. Positive Fractional Powers. Match the following functions to the appropriate graphs below:
$y = x^{1/2}, y = x^{1/3}$



Increasing for $0 < x < \infty$
Concave down for $0 < x < \infty$

Example 1. Find a formula for the power function that goes through the points $(4, 32)$ and $(\frac{1}{4}, \frac{1}{2})$.



Know: $f(x) = kx^p$

Given: $f(4) = 32$

$$f\left(\frac{1}{4}\right) = \frac{1}{2}$$

First, note that

$$f(4) = 32 \implies k \cdot 4^p = 32$$
$$f(1/4) = 1/2 \implies k \cdot (1/4)^p = 1/2$$

If we divide these two equations, we have

$$\frac{k \cdot 4^p}{k \cdot (1/4)^p} = \frac{32}{1/2} \implies \frac{4^p}{\left(\frac{1}{4}\right)^p} = \frac{32}{1} \cdot \frac{2}{1} \implies \frac{4^p}{1} \cdot \frac{4^p}{1^p} = 64$$
$$\implies 4^{2p} = 4^3$$
$$\implies 2p = 3,$$

which means that $p = 3/2$. Therefore,

$$k \cdot 4^p = 32 \implies k = \frac{32}{4^{3/2}} = 4,$$

and we see that our final answer is $f(x) = 4x^{3/2}$.

Definition.

1. A quantity y is called *directly proportional* to a power of x if $\underline{y = kx^n}$, where k and n are constants.
2. A quantity y is called *inversely proportional* to a power of x if $\underline{y = \frac{k}{x^n}}$, where k and n are constants.

Example 2. Write formulas that represent the following statements.

(a) The pressure, P, of a gas is inversely proportional to its volume, V.

$$P = \frac{k}{V}$$

(b) The work done, W, in stretching a spring is directly proportional to the square of the distance, d, that it is stretched.

$$W = kd^2$$

(c) The distance, d, of an object away from a planet is inversely proportional to the square root of the gravitational force, F, that the planet exerts on the object.

$$d = \frac{k}{\sqrt{F}}$$

Example 3. The blood circulation time (t) of a mammal is directly proportional to the 4th root of its mass (m). If a hippo having mass 2520 kilograms takes 123 seconds for its blood to circulate, how long will it take for the blood of a lion with body mass 180 kg to circulate?

Know: $t = km^{1/4}$

Given: $t = 123$ when $m = 2520$

Find: t when $m = 180$

First, using the given information, we have

$$t = km^{1/4} \implies 123 = k(2520)^{1/4} \implies \frac{123}{2520^{1/4}} = k \implies k \approx 17.36.$$

Therefore, our equation becomes $t = 17.36m^{1/4}$. To find the circulation time for the lion, we substitute $m = 180$ into our equation to obtain

$$t = 17.36(180)^{1/4} \approx 63.6 \text{ seconds},$$

which is our final answer.

Examples and Exercises

1. Find a formula for the power function $g(x)$ described by the table of values below. Be as accurate as you can with your rounding.

x	2	3	4	5
$g(x)$	4.5948	7.4744	10.5561	13.7973

Since g is a power function, we know that $g(x) = kx^p$, and we must solve for k and p. Using the table of values above, we see that

$$f(2) = 4.5948 \quad \text{and} \quad f(3) = 7.4744.$$

Therefore, we have

$$\begin{array}{rcl} f(2) &=& 4.5948 \\ f(3) &=& 7.4744 \end{array} \implies \begin{array}{rcl} k\cdot 2^p &=& 4.5948 \\ k\cdot 3^p &=& 7.4744 \end{array} \implies \frac{k\cdot 2^p}{k\cdot 3^p} = \frac{4.5948}{7.4744}$$

$$\implies \left(\frac{2}{3}\right)^p = 0.6147$$

$$\implies p\ln\left(\frac{2}{3}\right) \approx \ln(0.6147)$$

$$\implies p \approx \frac{\ln(0.6147)}{\ln(2/3)} \approx 1.2.$$

Thus,

$$k\cdot 2^p = 4.5948 \implies k\cdot 2^{1.2} = 4.5948 \implies k = \frac{4.5948}{2^{1.2}} \approx 2,$$

so our final answer is $f(x) = 2x^{1.2}$.

2. The pressure, P, exerted by a sample of hydrogen gas is inversely proportional to the volume, V, in the sample. A sample of hydrogen gas in a 2-liter container exerts a pressure of 1.5 atmospheres. How much pressure does the sample of gas exert if the size of the container is cut in half?

We are given that P is inversely proportional to V, so we know that

$$P = \frac{k}{V},$$

where k is a constant. We are also given that $P = 1.5$ atmospheres when $V = 2$ liters. Substituting this information into the above equation, we obtain

$$1.5 = \frac{k}{2} \qquad \Longrightarrow \qquad 3 = k,$$

so our equation becomes $P = 3/V$. Therefore, if the size of the container is halved, we have $V = 1$, and so sample of gas exerts a pressure of $P = 3/1 = 3$ atmospheres.

Sections 11.2 & 11.3 – Polynomials

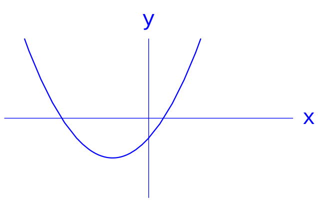
Definition. A *polynomial* is a function of the form

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a positive integer and $a_0, a_1, \ldots, a_n$ are all constants. The integer n is called the *degree* of the polynomial.

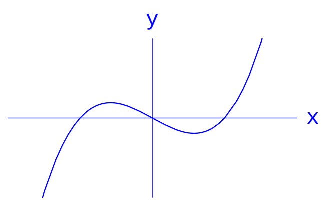
Fact 1. A polynomial of degree n can have at most $n-1$ "turnaround" points.

$n = 2$



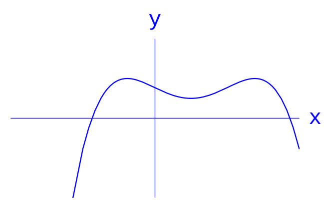
1 turnaround point

$n = 3$



2 turnaround points

$n = 4$

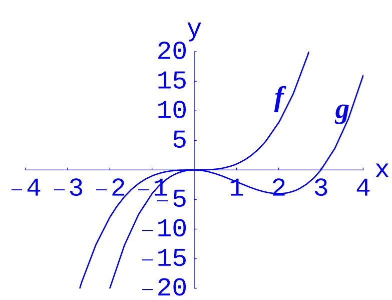


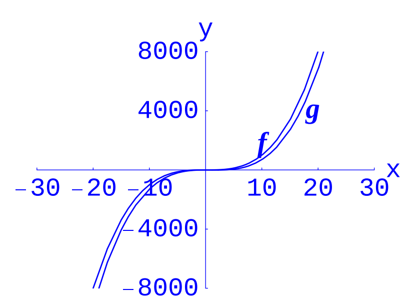
3 turnaround points

Fact 2. As $x \to \infty$ and $x \to -\infty$, the highest power of x "takes over." (**Note.** The symbol "$\to$" means "approaches.")

Example. Let $f(x) = x^3 - 3x^2$ and $g(x) = x^3$. First, graph these functions in the window $-4 \le x \le 4$ and $-20 \le y \le 20$. Then, graph them in the window $-30 \le x \le 30$ and $-8000 \le y \le 8000$. What do you notice?

The functions look different in the first window, but look very similar in the second window. This illustrates that, for large values of x, the "x^3" term that appears in the formula for $g(x)$ overpowers the "$-3x^2$" term, so that f and g both behave like $y = x^3$ for large values of x.



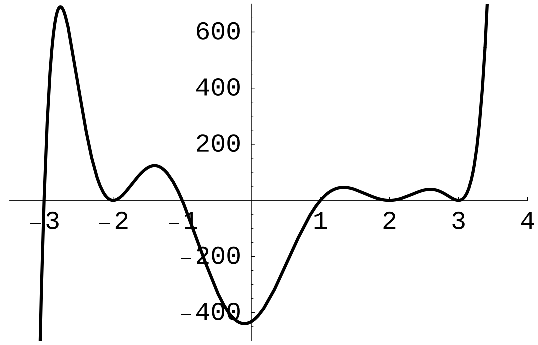


Fact 3. When a polynomial touches but does not cross the x axis at $x = a$, the factored form of the polynomial will have an even number of $(x - a)$ factors.

Example. Consider the polynomial

$$p(x) = (x + 3)(x + 2)^2(x + 1)(x - 1)(x - 2)^2(x - 3)^2,$$

whose graph is shown to the right.



Note that the factors $(x + 3)$, $(x + 1)$, and $(x - 1)$ are raised to the first power, and that the graph of f crosses through the x-axis at the corresponding numbers $x = -3$, $x = -1$, and $x = 1$. On the other hand, the factors $(x + 2)^2$, $(x - 2)^2$, $(x - 3)^2$ indicate that the graph of f simply touches the $x-$axis at the corresponding numbers $x = -2$, $x = 2$, and $x = 3$; it does not cross through the x-axis.

Definition. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial such that $a_n \neq 0$. Then the number a_n is called the *leading coefficient* of p, and the number a_0 is called the *constant coefficient* of p.

Example. Find the leading coefficient and the constant coefficient of each of the following.

1. $p(x) = 3x^4 - 5x^2 + 6x - 1$
2. $q(x) = x^2(x - 3)$
3. $r(x) = (2x - 3)^2(x + 4)$

1. The leading coefficient is 3 and the constant coefficient is -1.

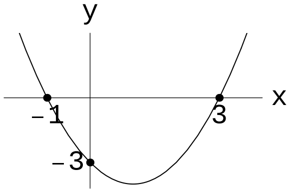
2. Expanding, we have
$$q(x) = x^2(x - 3) = x^3 - 3x^2.$$
Since $q(x)$ can be rewritten as $1x^3 - 3x^2 + 0x + 0$, we see that the leading coefficent is 1 and the constant coefficient is 0.

3. We have
$$\begin{aligned}(2x-3)^2(x+4) &= (2x-3)(2x-3)(x+4)\\ &= (2x)(2x)(x) + \texttt{ STUFF } + (-3)(-3)(4)\\ &= 4x^3 + \texttt{ STUFF } + 36,\end{aligned}$$
so the leading coefficient is 4 and the constant coefficient is 36.

Examples and Exercises

Each of the following gives the graph of a polynomial. Find a possible formula for each polynomial. In some cases, more than one answer is possible.

1.

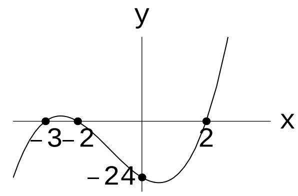


Since the function has zeroes at -1 and 3, we know that $f(x) = k(x+1)(x-3)$. Since

$$\begin{aligned} f(0) = -3 \quad &\Longrightarrow \quad k(0+1)(0-3) = -3 \\ &\Longrightarrow \quad k = 1, \end{aligned}$$

our final answer is $f(x) = (x+1)(x-3)$.

2.

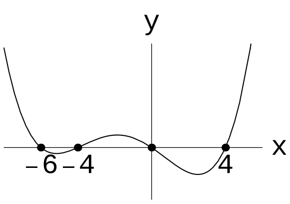


Here, we have $f(x) = k(x+3)(x+2)(x-2)$. Since

$$\begin{aligned} f(0) = -24 \quad &\Longrightarrow \quad -12k = -24 \\ &\Longrightarrow \quad k = 2, \end{aligned}$$

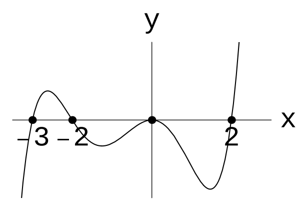
our answer is $f(x) = 2(x+3)(x+2)(x-2)$.

3.



In this case, $f(x) = kx(x+6)(x+4)(x-4)$. Note that the "x" term corresponds to the zero of the function at $x = 0$. We can see that $k > 0$ because $f(x)$ approaches ∞ as x approaches ∞, but we do not have enough information to determine a specific value for k. Therefore, any positive value of k will work, so one possible answer is $f(x) = x(x+6)(x+4)(x-4)$.

4.



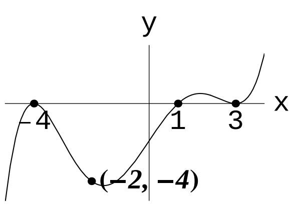
In this case, we have crossing points at $x = -3$, $x = -2$, and $x = 2$, and we have a touching point at $x = 0$. Therefore,

$$f(x) = kx^2(x+3)(x+2)(x-2).$$

Using similar logic as in Problem 3, we conclude that k can have any positive value, so one possible final answer is

$$f(x) = x^2(x+3)(x+2)(x-2).$$

5.

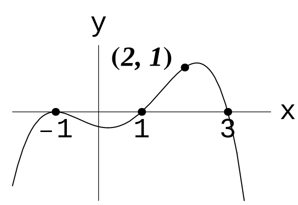


Here, we have $f(x) = k(x+4)^2(x-1)(x-3)^2$. Since

$$\begin{aligned} f(-2) = -4 \quad &\Longrightarrow \quad k(4)(-3)(25) = -4 \\ &\Longrightarrow \quad -300k = -4, \end{aligned}$$

we see that $k = 1/75$, so our final answer is $f(x) = \dfrac{1}{75}(x+4)^2(x-1)(x-3)^2$.

6.



Here, we have $f(x) = k(x+1)^2(x-1)(x-3)$. Since

$$\begin{aligned} f(2) = 1 \quad &\Longrightarrow \quad k(9)(1)(-1) = 1 \\ &\Longrightarrow \quad -9k = 1, \end{aligned}$$

we see that $k = -1/9$, so our final answer is $f(x) = -\dfrac{1}{9}(x+1)^2(x-1)(x-3)$.

For problems 7 through 10, answer each of the following questions. Use a graphing calculator where appropriate.

a. How many roots (zeros) does the polynomial have?
b. How many turning points does the polynomial have?

7. $y = 2x + 3$

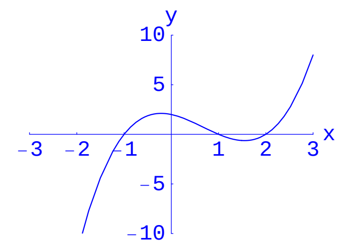
   We have $2x + 3 = 0 \implies 2x = -3 \implies x = -3/2$, so $x = -3/2$ is the only zero. Since this is the equation of a line, there are no turning points. Our answers are therefore: (a) 1, (b) none

8. $y = x^2 - x - 2$

   We have $x^2 - x - 2 = 0 \implies (x-2)(x+1) = 0$, so $x = 2$ and $x = -1$ are the zeros. Since this is the equation of a parabola, there is one turning point. Our answers are therefore: (a) 2, (b) 1

9. $y = x^3 - 2x^2 - x + 2$

   Referring to the graph to the right, we see that there are zeros at $x = -1$, $x = 1$ and $x = 2$. We also see two turnaround points, and we know that these must be the only turnaround points because a degree three polynomial can have at most two turnaround points. Our answers are therefore: (a) 3, (b) 2



10. $y = 5x^2 + 4$

    We have $5x^2 + 4 = 0 \implies x^2 = -4/5$, which has no solution because it is impossible to square any number x and get a negative result. Therefore, there are no zeros. Also, since this is the equation of a parabola, there is one turnaround point. Our answers are therefore: (a) none, (b) 1

For problems 11 through 15, answer the following questions about the given polynomial:

a. What is its degree?
b. What is its leading coefficient?
c. What is its constant coefficient?
d. What are the roots of the polynomial? First, give your answer(s) in exact form; then, give decimal approximations if appropriate.

11. $p(x) = x^2 - 3x - 28$

    First, note that

    $$x^2 - 3x - 28 = 0 \implies (x-7)(x+4) = 0 \implies x = 7 \text{ and } x = -4,$$

    so our answers are as follows:
    (a) 2, (b) 1, (c) -28, (d) 7 and -4

12. $p(x) = 8 - 7x$

    First, note that

    $$8 - 7x = 0 \implies 8 = 7x \implies x = 8/7,$$

    so our answers are as follows:
    (a) 1, (b) -7, (c) 8, (d) $8/7$

13. $p(x) = x(2 + 4x - x^2)$

    First, note that $x(2 + 4x - x^2) = 0$ implies that either $x = 0$ or $2 + 4x - x^2 = 0$. Therefore, $x = 0$ is one of the roots, and using the quadratic formula, the other two roots are

    $$\frac{-4 \pm \sqrt{4^2 - 4(-1)(2)}}{-2} = \frac{-4 \pm \sqrt{24}}{-2} = \frac{-4 \pm 2\sqrt{6}}{-2} = 2 \pm \sqrt{6}.$$

    Also, note that after expanding, $p(x) = -x^3 + 4x^2 + 2x + 0$, so the leading coefficient is -1 and the constant coefficient is 0. Our answers are therefore:
    (a) 3, (b) -1, (c) 0, (d) 0, $2 + \sqrt{6}$, $2 - \sqrt{6}$

14. $p(x) = 2x^2 + 4$

    Since $2x^2 + 4 = 0 \implies x^2 = -2$, this polynomial has no zeros. Our answers are therefore:
    (a) 2, (b) 2, (c) 4, (d) none

15. $p(x) = (x-3)(x+5)(x-37)(2x+4)x^2$

    First, note that if we multiplied out the factors of $p(x)$, we would get an expression of the form

    $$(x)(x)(x)(2x)x^2 + \texttt{ STUFF } + (-3)(5)(-37)(4)(0) \;=\; 2x^6 + \texttt{ STUFF } + 0.$$

    Therefore, our answers are:
    (a) 6, (b) 2, (c) 0, (d) 3, -5, 37, -2, 0

For problems 16 through 18, answer the following questions about the given polynomial:

a. What happens to the output values for extremely positive values of x?
b. What happens to the output values for extremely negative values of x?

16. $p(x) = -2x^3 + 6x - 2$

    For extremely positive and negative values of x, the leading term of the polynomial, $-2x^3$, takes over. Therefore, using the tables to the right, we conclude the following:

x	$-2x^3$
5	-250
10	-2000
100	-2000000

x	$-2x^3$
-5	250
-10	2000
-100	2000000

    (a) They approach $-\infty$, (b) They approach ∞.

17. $p(x) = 2x - x^2$

    By examining the behavior of $-x^2$ for extremely positive and negative values of x using methods similar to those in Problem 16 above, we obtain:
    (a) They approach $-\infty$, (b) They approach $-\infty$.

18. $p(x) = -x^6 - x - 2$

    By examining the behavior of $-x^6$ for extremely positive and negative values of x using methods similar to those in Problem 16 above, we obtain:
    (a) They approach $-\infty$, (b) They approach $-\infty$.

19. For each of the following, give a formula for a polynomial with the indicated properties.

    a. A sixth degree polynomial with 6 roots.

    Note that the polynomial function $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)$ has $1, 2, 3, 4, 5$, and 6 as its roots and is a 6th degree polynomial. Therefore, one of many possible answers is

    $$f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6).$$

    b. A sixth degree polynomial with no roots.

    Note that if we let $f(x) = x^6 + 1$ and try to solve the equation $f(x) = 0$, we obtain

    $$x^6 + 1 = 0 \quad\implies\quad x^6 = -1,$$

    which has no solution since no number x raised to the 6th power will ever be negative. Therefore, $x^6 + 1$ is a 6th degree polynomial with no roots, so

    $$f(x) = x^6 + 1$$

    is one of many possible answers.

Sections 11.4 & 11.5 – Rational Functions

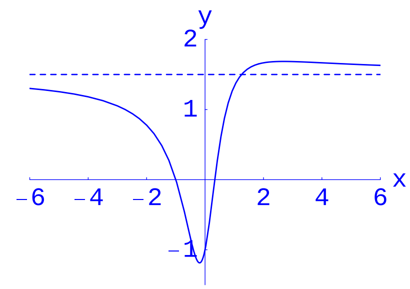
Definition. A *rational function* is a function $r(x)$ of the form $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. In other words, a rational function is a polynomial divided by a polynomial.

Example 1. Let $f(x) = \dfrac{3x^2+2x-1}{2x^2+1}$. First, fill in the table to the right for the function $f(x)$. Then, sketch a graph of $f(x)$ in the space below.

x	1	10	100	1000	10000
$f(x)$	1.33	1.59	1.51	1.501	1.5001

Referring to the graph of f to the right, it appears that $y = 1.5$ is a horizontal asymptote. We make the following observations:

As $x \to \infty$, $\quad f(x) \to 1.5 \quad$ (see table above)
As $x \to -\infty$, $\quad f(x) \to 1.5$



Note: For any function f, if $f(x) \to L$ as $x \to \infty$ or $x \to -\infty$, then the line $y = L$ is a horizontal asymptote of f.

Example 2. Algebraically check each of the following for horizontal asymptotes.

(a) $f(x) = \dfrac{3x^2+2x-1}{2x^2+1}$ (b) $g(x) = \dfrac{2x+4}{2x^2+1}$ (c) $h(x) = \dfrac{x^6+5x^3-2x^2+1}{x^4+2}$

(a) Observe that, for large values of x,

$$\frac{3x^2+2x-1}{2x^2+1} \approx \frac{3x^2}{2x^2} = \frac{3}{2},$$

so we conclude that $y = 3/2$ is a horizontal asymptote of f.

(b) Observe that, for large values of x,

$$\frac{2x+4}{2x^2+1} \approx \frac{2x}{2x^2} = \frac{1}{x},$$

and $1/x$ approaches 0 as x gets large in magnitude. Therefore, $y = 0$ is a horizontal asymptote of g.

(c) Observe that, for large values of x,

$$\frac{x^6+5x^3-2x^2+1}{x^4+2} \approx \frac{x^6}{x^4} = x^2,$$

and x^2 approaches ∞ (which is not a real number) as x gets large in magnitude. Therefore, h does not have any horizontal asymptotes.

Finding horizontal asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function.

1. If the degree of $p(x)$ equals the degree of $q(x)$, then $f(x)$ has a horizontal asymptote at
$$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}.$$
2. If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is a horizontal asymptote.
3. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $f(x)$ has no horizontal asymptote.

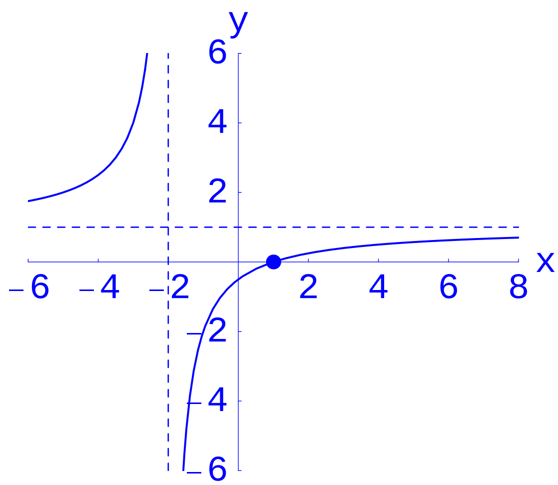
Example 3. Let $f(x) = \dfrac{x-1}{x+2}$. What happens when $x = 1$? What happens when $x = -2$? Graph this function for $-5 \le x \le 5$ and $-5 \le y \le 5$.

x	-1.9	-1.99	-1.999	-1.9999
$f(x)$	-29	-299	-2999	-29999

As $x \to -2^+$, $f(x) \to -\infty$

x	-2.1	-2.01	-2.001	-2.0001
$f(x)$	31	301	3001	30001

As $x \to -2^-$, $f(x) \to \infty$



Observations from Graph:

- $x = -2$ is a vertical asymptote.
- $y = 1$ is a horizontal asymptote.
- f has an x-intercept at $x = 1$.

Finding vertical asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. To find vertical asymptotes, look at places where the denominator $q(x) = 0$.

Examples and Exercises

For each of the following rational functions, find all horizontal and vertical asymptotes (if there are any), all x-intercepts (if there are any), and the y-intercept. Find exact and approximate values when possible.

1. $f(x) = \dfrac{3x-4}{7x+1}$

$$\begin{aligned}
3x-4=0 \quad &\Longrightarrow \quad x = 4/3 && \longleftarrow \quad \texttt{x-intercept} \\
7x+1=0 \quad &\Longrightarrow \quad x=-1/7 && \longleftarrow \quad \texttt{vertical asymptote} \\
& y = 3/7 && \longleftarrow \quad \texttt{horizontal asymptote} \\
& f(0) = -4 && \longleftarrow \quad \texttt{y-intercept}
\end{aligned}$$

2. $f(x) = \dfrac{x^2+10x+24}{x^2-2x+1}$

$$\begin{aligned}
x^2+10x+24=0 \quad \Longrightarrow \quad (x+6)(x+4)=0 \quad &\Longrightarrow \quad x=-6,\ x=-4 && \longleftarrow \quad \texttt{x-intercepts} \\
x^2-2x+1=0 \quad \Longrightarrow \quad (x-1)^2=0 \quad &\Longrightarrow \quad x=1 && \longleftarrow \quad \texttt{vertical asymptote} \\
& y=1 && \longleftarrow \quad \texttt{horizontal asymptote} \\
& f(0)=24 && \longleftarrow \quad \texttt{y-intercept}
\end{aligned}$$

3. $f(x) = \dfrac{2x^3+1}{x^2+x}$

$$\begin{aligned}
2x^3+1=0 \quad \Longrightarrow \quad x^3=-1/2 \quad &\Longrightarrow \quad x=\sqrt[3]{-1/2} && \longleftarrow \quad \texttt{x-intercept} \\
x^2+x=0 \quad \Longrightarrow \quad x(x+1)=0 \quad &\Longrightarrow \quad x=0,\ x=-1 && \longleftarrow \quad \texttt{vertical asymptotes} \\
&\texttt{(no horizontal asymptotes)} \\
&\texttt{(no y-intercept)}
\end{aligned}$$

4. $f(x) = \dfrac{(x^2-4)(x^2+1)}{x^6}$

$$\begin{aligned}
(x^2-4)(x^2+1)=0 \quad \Longrightarrow \quad (x-2)(x+2)(x^2+1)=0 \quad &\Longrightarrow \quad x=2,\ x=-2 && \longleftarrow \quad \texttt{x-intercepts} \\
x^6=0 \quad &\Longrightarrow \quad x=0 && \longleftarrow \quad \texttt{vertical asymptote} \\
& y=0 && \longleftarrow \quad \texttt{horizontal asymptote} \\
&\texttt{(no y-intercept)}
\end{aligned}$$

5. $f(x) = \dfrac{2x+1}{6x^2+31x-11}$

$$\begin{aligned}
2x+1=0 \quad &\Longrightarrow \quad x=-1/2 && \longleftarrow \quad \texttt{x-intercept} \\
6x^2+31x-11=(3x-1)(2x+11)=0 \quad &\Longrightarrow \quad x=1/3,\ x=-11/2 && \longleftarrow \quad \texttt{vertical asymptotes} \\
& y=0 && \longleftarrow \quad \texttt{horizontal asymptote} \\
& f(0)=-1/11 && \longleftarrow \quad \texttt{y-intercept}
\end{aligned}$$

6. $f(x) = \dfrac{2x^2 - c}{(x-c)(3x+d)}$, where c and d are constants, and $c \neq 0$.

$$\begin{array}{rcrcl}
2x^2 - c = 0 & \Longrightarrow & x^2 = c/2 \quad \Longrightarrow \quad x = \pm\sqrt{c/2} & \longleftarrow & \texttt{x-intercepts (if $c>0$)} \\
(x-c)(3x+d) & \Longrightarrow & x = c, \;\; x = -d/3 & \longleftarrow & \texttt{vertical asymptotes} \\
 & & y = 2/3 & \longleftarrow & \texttt{horizontal asymptote} \\
 & & f(0) = 1/d & \longleftarrow & \texttt{y-intercept (if $d \neq 0$)}
\end{array}$$

7. $f(x) = \dfrac{1}{x-3} + \dfrac{1}{x-5} = \dfrac{x-5}{(x-3)(x-5)} + \dfrac{x-3}{(x-3)(x-5)} = \dfrac{2x-8}{(x-3)(x-5)}$

Hint: First, find a common denominator.

$$\begin{array}{rcrcl}
2x - 8 = 0 & \Longrightarrow & x = 4 & \longleftarrow & \texttt{x-intercept} \\
(x-3)(x-5) & \Longrightarrow & x = 3, \;\; x = 5 & \longleftarrow & \texttt{vertical asymptotes} \\
 & & y = 0 & \longleftarrow & \texttt{horizontal asymptote} \\
 & & f(0) = -8/15 & \longleftarrow & \texttt{y-intercept}
\end{array}$$

8. $f(x) = \dfrac{x^5 - 2x^4 - 9x + 18}{8x^3 + 2x^2 - 3x} = \dfrac{x^4(x-2) - 9(x-2)}{x(8x^2 + 2x - 3)} = \dfrac{(x^4 - 9)(x-2)}{x(8x^2 + 2x - 3)} = \dfrac{(x^2-3)(x^2+3)(x-2)}{x(2x-1)(4x+3)}$

Hint: $x^5 - 2x^4 - 9x + 18 = x^4(x-2) - 9(x-2)$

$$\begin{array}{rcrcl}
(x-2)(x^2-3)(x^2+3) = 0 & \Longrightarrow & x = 2, \;\; x = -\sqrt{3}, \;\; x = \sqrt{3} & \longleftarrow & \texttt{x-intercepts} \\
x(2x-1)(4x+3) = 0 & \Longrightarrow & x = 0, \;\; x = 1/2, \;\; x = -3/4 & \longleftarrow & \texttt{vertical asymptotes}
\end{array}$$

(no horizontal asymptotes)

(no y-intercept)

9. $f(x) = \dfrac{1}{x-1} + \dfrac{2}{x+2} + 3 = \dfrac{x+2}{(x-1)(x+2)} + \dfrac{2(x-1)}{(x-1)(x+2)} + \dfrac{3(x-1)(x+2)}{(x-1)(x+2)} = \dfrac{3x^2 + 6x - 6}{(x-1)(x+2)}$

Hint: First, find a common denominator.

$$\begin{array}{rcl}
3x^2 + 6x - 6 = 0 & \Longrightarrow & x^2 + 2x - 2 = 0 \\
 & \Longrightarrow & x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-2)}}{2} \\
 & \Longrightarrow & x = \dfrac{-2 \pm 2\sqrt{3}}{2}
\end{array}$$

$$\begin{array}{rcrcl}
 & \Longrightarrow & x = -1 - \sqrt{3}, \;\; x = -1 + \sqrt{3} & \longleftarrow & \texttt{x-intercepts} \\
(x-1)(x+2) = 0 & \Longrightarrow & x = 1, \;\; x = -2 & \longleftarrow & \texttt{vertical asymptotes} \\
 & & y = 3 & \longleftarrow & \texttt{horizontal asymptote} \\
 & & f(0) = 3 & \longleftarrow & \texttt{y-intercept}
\end{array}$$