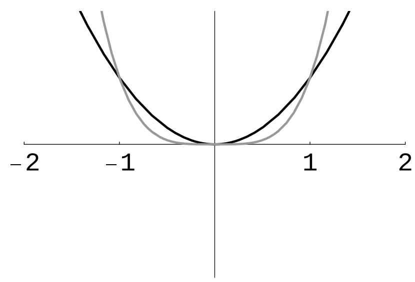
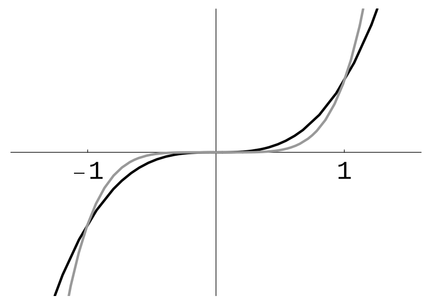
Section 11.1 – Power Functions and Proportionality

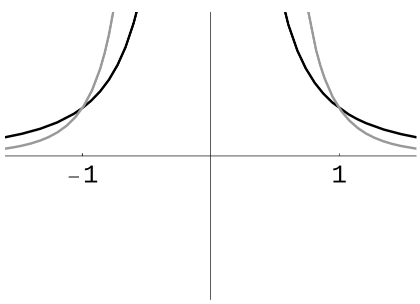
Definition. A *power function* is a function of the form $f(x) = kx^p$, where k and p are constants.

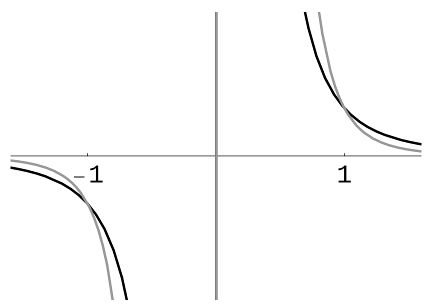
I. Positive Integer Powers. Match the following functions to the appropriate graphs below:
$y = x^2, y = x^3, y = x^4, y = x^5$



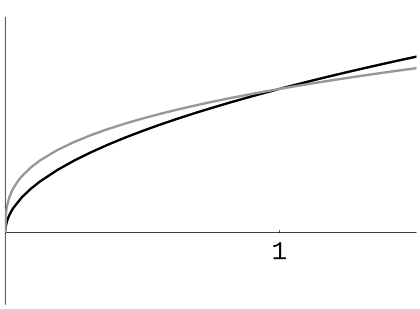


II. Negative Integer Powers. Match the following functions to the appropriate graphs below:
$y = x^{-2}, y = x^{-3}, y = x^{-4}, y = x^{-5}$





III. Positive Fractional Powers. Match the following functions to the appropriate graphs below:
$y = x^{1/2}, y = x^{1/3}$



Example 1. Find a formula for the power function that goes through the points $(4, 32)$ and $(\frac{1}{4}, \frac{1}{2})$.

Definition.

1. A quantity y is called *directly proportional* to a power of x if _______________, where k and n are constants.
2. A quantity y is called *inversely proportional* to a power of x if _______________, where k and n are constants.

Example 2. Write formulas that represent the following statements.

(a) The pressure, P, of a gas is inversely proportional to its volume, V.

(b) The work done, W, in stretching a spring is directly proportional to the square of the distance, d, that it is stretched.

(c) The distance, d, of an object away from a planet is inversely proportional to the square root of the gravitational force, F, that the planet exerts on the object.

Example 3. The blood circulation time (t) of a mammal is directly proportional to the 4th root of its mass (m). If a hippo having mass 2520 kilograms takes 123 seconds for its blood to circulate, how long will it take for the blood of a lion with body mass 180 kg to circulate?

Examples and Exercises

1. Find a formula for the power function $g(x)$ described by the table of values below. Be as accurate as you can with your rounding.

x	2	3	4	5
$g(x)$	4.5948	7.4744	10.5561	13.7973

2. The pressure, P, exerted by a sample of hydrogen gas is inversely proportional to the volume, V, in the sample. A sample of hydrogen gas in a 2-liter container exerts a pressure of 1.5 atmospheres. How much pressure does the sample of gas exert if the size of the container is cut in half?

Sections 11.2 & 11.3 – Polynomials

Definition. A *polynomial* is a function of the form

$$y = p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a positive integer and $a_0, a_1, \ldots, a_n$ are all constants. The integer n is called the *degree* of the polynomial.

Fact 1. A polynomial of degree n can have at most $n - 1$ "turnaround" points.

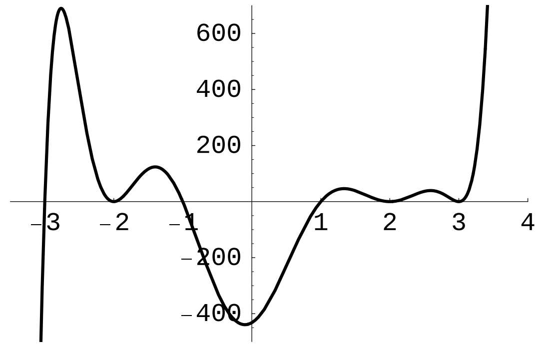
Fact 2. As $x \to \infty$ and $x \to -\infty$, the highest power of x "takes over." (**Note.** The symbol "$\to$" means "approaches.")

Fact 3. When a polynomial touches but does not cross the x axis at $x = a$, the factored form of the polynomial will have an even number of $(x - a)$ factors.

Example. Consider the polynomial

$$p(x) = (x + 3)(x + 2)^2(x + 1)(x - 1)(x - 2)^2(x - 3)^2,$$

whose graph is shown to the right.



Definition. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial such that $a_n \neq 0$. Then the number a_n is called the *leading coefficient* of p, and the number a_0 is called the *constant coefficient* of p.

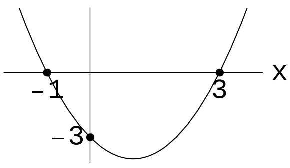
Example. Find the leading coefficient and the constant coefficient of each of the following.

1. $p(x) = 3x^4 - 5x^2 + 6x - 1$
2. $q(x) = x^2(x - 3)$
3. $r(x) = (2x - 3)^2(x + 4)$

Examples and Exercises

Each of the following gives the graph of a polynomial. Find a possible formula for each polynomial. In some cases, more than one answer is possible.

1.



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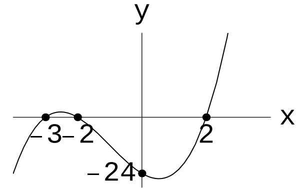

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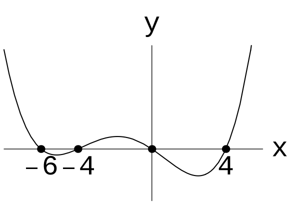

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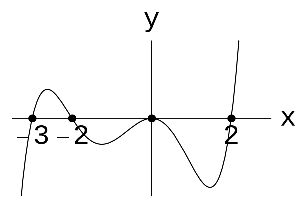

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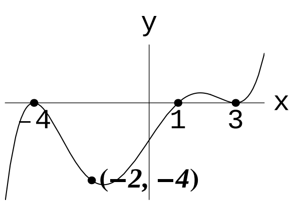

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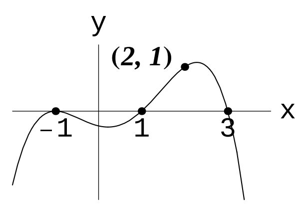

For problems 7 through 10, answer each of the following questions. Use a graphing calculator where appropriate.

a. How many roots (zeros) does the polynomial have?
b. How many turning points does the polynomial have?

7. $y = 2x + 3$

8. $y = x^2 - x - 2$

9. $y = x^3 - 2x^2 - x + 2$

10. $y = 5x^2 + 4$

For problems 11 through 15, answer the following questions about the given polynomial:

a. What is its degree?
b. What is its leading coefficient?
c. What is its constant coefficient?
d. What are the roots of the polynomial? First, give your answer(s) in exact form; then, give decimal approximations if appropriate.

11. $p(x) = x^2 - 3x - 28$

12. $p(x) = 8 - 7x$

13. $p(x) = x(2 + 4x - x^2)$

14. $p(x) = 2x^2 + 4$

15. $p(x) = (x-3)(x+5)(x-37)(2x+4)x^2$

For problems 16 through 18, answer the following questions about the given polynomial:

a. What happens to the output values for extremely positive values of x?
b. What happens to the output values for extremely negative values of x?

16. $p(x) = -2x^3 + 6x - 2$

17. $p(x) = 2x - x^2$

18. $p(x) = -x^6 - x - 2$

19. For each of the following, give a formula for a polynomial with the indicated properties.

    a. A sixth degree polynomial with 6 roots.

    b. A sixth degree polynomial with no roots.

Sections 11.4 & 11.5 – Rational Functions

Definition. A *rational function* is a function $r(x)$ of the form $r(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. In other words, a rational function is a polynomial divided by a polynomial.

Example 1. Let $f(x) = \dfrac{3x^2 + 2x - 1}{2x^2 + 1}$. First, fill in the table to the right for the function $f(x)$. Then, sketch a graph of $f(x)$ in the space below.

x	1	10	100	1000	10000
$f(x)$					

Example 2. Algebraically check each of the following for horizontal asymptotes.

(a) $f(x) = \dfrac{3x^2 + 2x - 1}{2x^2 + 1}$ (b) $g(x) = \dfrac{2x + 4}{2x^2 + 1}$ (c) $h(x) = \dfrac{x^6 + 5x^3 - 2x^2 + 1}{x^4 + 2}$

Finding horizontal asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function.

1. If the degree of $p(x)$ <u>equals</u> the degree of $q(x)$, then $f(x)$ has a horizontal asymptote at

$$y = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}.$$

2. If the degree of $p(x)$ is <u>less than</u> the degree of $q(x)$, then $y = 0$ is a horizontal asymptote.

3. If the degree of $p(x)$ is <u>greater than</u> the degree of $q(x)$, then $f(x)$ has no horizontal asymptote.

Example 3. Let $f(x) = \dfrac{x-1}{x+2}$. What happens when $x = 1$? What happens when $x = -2$? Graph this function for $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$.

x	-1.9	-1.99	-1.999	-1.9999
$f(x)$				

x	-2.1	-2.01	-2.001	-2.0001
$f(x)$				

Finding vertical asymptotes. Let $f(x) = \frac{p(x)}{q(x)}$ be a rational function. To find vertical asymptotes, look at places where the denominator $q(x) = 0$.

Examples and Exercises

For each of the following rational functions, find all horizontal and vertical asymptotes (if there are any), all x-intercepts (if there are any), and the y-intercept. Find exact and approximate values when possible.

1. $f(x) = \dfrac{3x-4}{7x+1}$

2. $f(x) = \dfrac{x^2+10x+24}{x^2-2x+1}$

3. $f(x) = \dfrac{2x^3+1}{x^2+x}$

4. $f(x) = \dfrac{(x^2-4)(x^2+1)}{x^6}$

5. $f(x) = \dfrac{2x+1}{6x^2+31x-11}$

6. $f(x) = \dfrac{2x^2 - c}{(x-c)(3x+d)}$, where c and d are constants, and $c \neq 0$.

7. $f(x) = \dfrac{1}{x-3} + \dfrac{1}{x-5}$
   Hint: First, find a common denominator.

8. $f(x) = \dfrac{x^5 - 2x^4 - 9x + 18}{8x^3 + 2x^2 - 3x}$
   Hint: $x^5 - 2x^4 - 9x + 18 = x^4(x-2) - 9(x-2)$

9. $f(x) = \dfrac{1}{x-1} + \dfrac{2}{x+2} + 3$
   Hint: First, find a common denominator.