Section 10.1 – Function Composition

The function $h(t) = f(g(t))$ is called the *composition of f with g*. The function h is defined by using the output of the function g as the input of f.

Example 1. Complete the table below.

t	0	1	2	3
$f(t)$	2	3	1	1
$g(t)$	3	1	2	0
$f(g(t))$	1	3	1	2
$g(f(t))$	2	0	1	1

$$\begin{aligned} f(g(0)) &= f(3) &= 1 \\ g(f(0)) &= g(2) &= 2 \\ f(g(2)) &= f(2) &= 1 \\ g(f(2)) &= g(1) &= 1 \end{aligned}$$

Example 2. Let $f(x) = x^2 - 1$, $g(x) = \dfrac{2x^2}{x-1}$, and $p(x) = \sqrt{x}$. Find and simplify each of the following.

(a) $g(f(x))$ (b) $p(g(x^2))$ (c) $\dfrac{f(x+h) - f(x)}{h}$

(a) $g(f(x)) = g(x^2 - 1) = \dfrac{2(x^2-1)^2}{(x^2-1)-1} = \dfrac{2(x^4 - 2x^2 + 1)}{x^2 - 2} = \dfrac{2x^4 - 4x^2 + 2}{x^2 - 2}$.

(b) $p(g(x^2)) = p\left(\dfrac{2(x^2)^2}{x^2 - 1}\right) = \sqrt{\dfrac{2x^4}{x^2-1}} = \dfrac{x^2\sqrt{2}}{\sqrt{x^2-1}}$.

(c) First, note that $f(x) = x^2 - 1$, so $f(x+h) = (x+h)^2 - 1 = x^2 + 2xh + h^2 - 1$. Therefore, we have

$$\begin{aligned} \frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - 1) - (x^2 - 1)}{h} &= \frac{x^2 + 2xh + h^2 - 1 - x^2 + 1}{h} \\ &= \frac{2xh + h^2}{h} \\ &= \frac{h(2x+h)}{h} \\ &= 2x + h, \end{aligned}$$

so our final answer is $2x + h$.

Example 3. For the function $f(x) = (x^3+1)^2$, find functions $u(x)$ and $v(x)$ such that $f(x) = u(v(x))$.

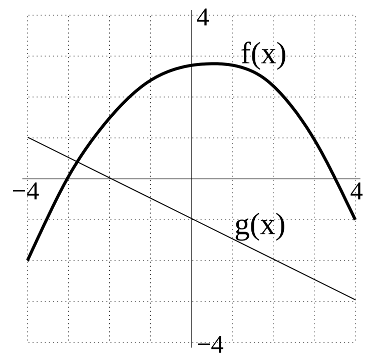
First, we let $v(x) = x^3 + 1$, the "inside" function. Then $f(x) = (x^3+1)^2 = (v(x))^2$, so we see that the function $v(x)$ is being squared to obtain $f(x)$. Therefore, $u(x) = x^2$. To see that our answer is right, note that

$$u(v(x)) = (v(x))^2 = (x^3+1)^2 = f(x).$$

Thus, our answers are $u(x) = x^2$ and $v(x) = x^3 + 1$.

Examples and Exercises

1. Given to the right are the graphs of two functions, f and g. Use the graphs to estimate each of the following.



(a) $g(f(0)) =$ $\underline{\ -2.3\ }$

$g(f(0)) = g(2.8) = -2.3$

(b) $f(g(0)) =$ $\underline{2.4}$

$f(g(0)) = f(-1) = 2.4$

(c) $f(g(3)) =$ $\underline{\ 0.9\ }$

$f(g(3)) = f(-2.5) = 0.9$

(d) $g(g(4)) =$ $\underline{\ 0.5\ }$

$g(g(4)) = g(-3) = 0.5$

(e) $f(f(1)) =$ $\underline{\ 1.3\ }$

$f(f(1)) = f(2.8) = 1.3$

2. For each of the following functions $f(x)$, find functions $u(x)$ and $v(x)$ such that $f(x) = u(v(x))$.

(a) $\sqrt{1+x}$

Let $v(x) = 1 + x$ be the "inside" function. Then $\sqrt{1+x} = \sqrt{v(x)}$, so we need to take the square root of $v(x)$ to get $f(x) = \sqrt{1+x}$. Therefore, $u(x) = \sqrt{x}$. To check our answer, note that

$$u(v(x)) = u(1+x) = \sqrt{1+x} = f(x),$$

so our final answer is $u(x) = \sqrt{x}$ and $v(x) = 1 + x$.

(b) $\sin(x^3+1)\cos(x^3+1)$

Let $v(x) = x^3 + 1$ be the "inside" function. Then $\sin(x^3+1)\cos(x^3+1) = \sin(v(x))\cos(v(x))$, so we need to substitute $v(x)$ into the function $u(x) = \sin x \cos x$ to get $f(x) = \sin(x^3+1)\cos(x^3+1)$. To check our answer, note that

$$u(v(x)) = u(x^3+1) = \sin(x^3+1)\cos(x^3+1) = f(x),$$

so our final answer is $u(x) = \sin x \cos x$ and $v(x) = x^3 + 1$.

(c) 3^{2x+1}

Let $v(x) = 2x+1$, so that $3^{2x+1} = 3^{v(x)}$. Therefore, if we choose $u(x) = 3^x$, we have

$$u(v(x)) = u(2x+1) = 3^{2x+1}$$

as desired. Our final answers are therefore $u(x) = 3^x$ and $v(x) = 2x+1$.

(d) $\dfrac{1}{1+\frac{2}{x}}$

Let $v(x) = 2/x$ and $u(x) = 1/(1+x)$. Then

$$u(v(x)) = u\left(\frac{2}{x}\right) = \frac{1}{1+\frac{2}{x}}$$

as desired. Our final answers are therefore $u(x) = 1/(1+x)$ and $v(x) = 2/x$.

3. Let $f(x) = \dfrac{1}{1+2x}$.

(a) Solve $f(x+1) = 4$ for x.

We have

$$\begin{aligned} f(x+1) = 4 \quad \implies \quad \frac{1}{1+2(x+1)} = 4 \quad \implies \quad \frac{1}{2x+3} = 4 \quad &\implies \quad 1 = 4(2x+3) \\ &\implies \quad 1 = 8x+12 \\ &\implies \quad 8x = -11, \end{aligned}$$

so our final answer is $x = -11/8$.

(b) Solve $f(x) + 1 = 4$ for x.

We have

$$\begin{aligned} f(x)+1 = 4 \quad \implies \quad f(x) = 3 \quad \implies \quad \frac{1}{1+2x} = 3 \quad &\implies \quad 1 = 3(1+2x) \\ &\implies \quad 1 = 3+6x \\ &\implies \quad 6x = -2, \end{aligned}$$

so our answer is $x = -1/3$.

(c) Calculate $f(f(x))$ and simplify your answer.

We have

$$\begin{aligned} f(f(x)) \quad = \quad f\left(\frac{1}{1+2x}\right) \quad = \quad \frac{1}{1+2\left(\frac{1}{1+2x}\right)} \quad &= \quad \frac{1}{1+\frac{2}{1+2x}} \\ &= \quad \frac{1}{\frac{1+2x}{1+2x}+\frac{2}{1+2x}} \\ &= \quad \frac{1}{\frac{1+2x+2}{1+2x}} \\ &= \quad \frac{1}{1}\cdot\frac{1+2x}{3+2x}, \end{aligned}$$

so our final answer is $\dfrac{1+2x}{3+2x}$.

4. For each of the following functions, calculate

$$\frac{f(x+h)-f(x)}{h}$$

and simplify your answers.

(a) $f(x)=x^2+2x+1$ (b) $f(x)=\frac{1}{x}$ (c) $f(x)=3x+1$

(a)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2+2(x+h)+1-(x^2+2x+1)}{h} = \frac{x^2+2xh+h^2+2x+2h+1-x^2-2x-1}{h}\\ &= \frac{2xh+h^2+2h}{h}\\ &= \frac{h(2x+h+2)}{h},\end{aligned}$$

so after canceling the factor of h, our final answer is $2x+h+2$.

(b)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h}\\ &= \frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}}\\ &= \frac{-h}{x(x+h)}\cdot\frac{1}{h},\end{aligned}$$

so after canceling the factor of h, our final answer is $\frac{-1}{x(x+h)}$.

(c)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{3(x+h)+1-(3x+1)}{h} = \frac{3x+3h+1-3x-1}{h}\\ &= \frac{3h}{h}\\ &= 3,\end{aligned}$$

so our final answer is simply 3.

Section 10.2 – Invertibility and Properties of Inverse Functions

Definition. Suppose $Q = f(t)$ is a function with the property that each value of Q determines exactly one value of t. Then f has an *inverse function*, f^{-1}, and

$$f^{-1}(Q) = t \quad \text{if and only if} \quad Q = f(t).$$

If a function has an inverse, it is said to be *invertible*.

Example 1. Given below are values for a function $Q = f(t)$. Fill in the corresponding table for $t = f^{-1}(Q)$.

t	0	1	2	3	4
$f(t)$	2	5	7	8	11

Q	2	5	7	8	11
$f^{-1}(Q)$	0	1	2	3	4

Observation:

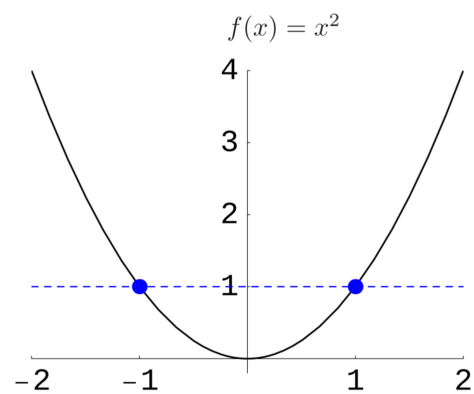
$$f^{-1}(f(2)) = f^{-1}(7) = 2$$
$$f(f^{-1}(5)) = f(1) = 5$$

Comment: In general, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Question. Does the function $f(x) = x^2$ have an inverse function?

No, because (for example) $f(1) = 1$ and $f(-1) = 1$. Therefore, if there were a function f^{-1}, we would have $f^{-1}(1)$ equal to both 1 and -1, which is impossible.

Comment: Note that this example, when interpreted graphically, illustrates that $f(x) = x^2$ fails the Horizontal Line Test.



Horizontal Line Test. A function f has an inverse function if and only if the graph of f intersects any horizontal line *at most once*. In other words, if any horizontal line touches the graph of f in more than one place, then f is not invertible.

Example 2. Suppose $B = f(t) = 5(1.04)^t$, where B is the balance in a bank account, in thousands of dollars, after t years.

(a) Find a formula for the inverse function of f.

We are given B as a function of t, so finding the inverse amounts to finding t as a function of B. We have

$$\begin{aligned} B &= 5(1.04)^t && \longleftarrow B \text{ as a function of } t \\ \frac{B}{5} &= 1.04^t \\ \ln(B/5) &= t\ln 1.04 \\ t &= \frac{\ln(B/5)}{\ln 1.04} && \longleftarrow t \text{ as a function of } B \end{aligned}$$

Therefore, the formula for the inverse function is $f^{-1}(B) = \dfrac{\ln(B/5)}{\ln 1.04}$.

(b) Compute each of the following and interpret them practically: (i) $f(20)$ (ii) $f^{-1}(20)$

(i) $f(20) = 5(1.04)^{20} \approx 10.96$ thousand dollars is the amount of money in the account after 20 years.

(ii) Using the answer to part (a) above, we have

$$f^{-1}(20) = \frac{\ln(20/5)}{\ln 1.04} \approx 35.35 \text{ years.}$$

Therefore, $f^{-1}(20) = 35.35$ years is the amount of time it takes until the account has 20 thousand dollars in it.

Examples and Exercises

1. Find a formula for the inverse function of each of the following functions.

   (a) $f(x) = \dfrac{x-1}{x+1}$

   `We have`

$$
\begin{aligned}
y = \frac{x-1}{x+1} \quad \Longrightarrow \quad y(x+1) = x-1 \quad &\Longrightarrow \quad yx + y = x - 1 \\
&\Longrightarrow \quad yx - x = -1 - y \\
&\Longrightarrow \quad x(y-1) = -1 - y,
\end{aligned}
$$

   `so` $x = \dfrac{-1-y}{y-1}$, `and our answer is therefore` $f^{-1}(y) = \dfrac{-1-y}{y-1}$.

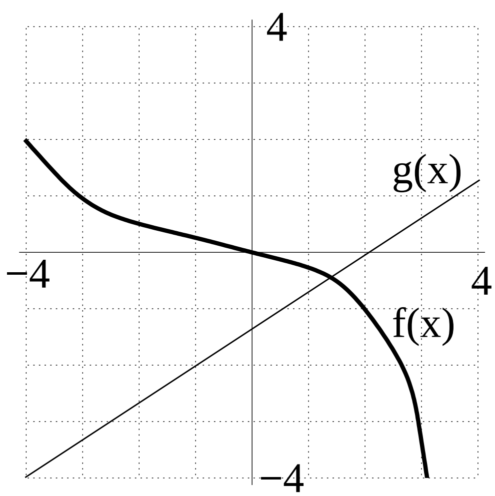
   (b) $g(x) = \ln(3-x)$

   `We have`

$$
\begin{aligned}
y = \ln(3-x) \quad \Longrightarrow \quad e^y = e^{\ln(3-x)} \quad &\Longrightarrow \quad e^y = 3 - x \\
&\Longrightarrow \quad x = 3 - e^y,
\end{aligned}
$$

   `so our answer is` $g^{-1}(y) = 3 - e^y$.

2. Given to the right is the graph of the functions $f(x)$ and $g(x)$. Use the function to estimate each of the following.



   (a) $f(2) =$ <u>-1</u>

   (b) $f^{-1}(2) =$ <u>-4</u>

   (c) $f^{-1}(g(-1)) =$ <u>2.6</u>

   $f^{-1}(g(-1)) \approx f^{-1}(-2) \approx 2.6$

   (d) $g^{-1}(f(3)) =$ <u>-2.7</u>

   $g^{-1}(f(3)) \approx g^{-1}(-3.2)$
   ≈ -2.7

   (e) Rank the following quantities in order from smallest to largest: $f(1),\ f(-2),\ f^{-1}(1),\ f^{-1}(-2),\ 0$

$$f^{-1}(1) \;<\; f(1) \;<\; 0 \;<\; f(-2) \;<\; f^{-1}(-2)$$

3. Let $f(x) = 10e^{(x-1)/2}$ and $g(x) = 2\ln x - 2\ln 10 + 1$. Show that $g(x)$ is the inverse function of $f(x)$.

We have

$$\begin{aligned} f(g(x)) \;=\; f(2\ln x - 2\ln 10 + 1) \;&=\; 10\mathrm{e}^{(2\ln x - 2\ln 10 + 1 - 1)/2} \\ &=\; 10\mathrm{e}^{2(\ln x - \ln 10)/2} \\ &=\; 10\mathrm{e}^{\ln x - \ln 10} \\ &=\; 10\mathrm{e}^{\ln(x/10)} \\ &=\; 10 \cdot (x/10) \;=\; x, \end{aligned}$$

so $f(g(x)) = x$ for all x. Similarly, we have

$$\begin{aligned} g(f(x)) &= 2\ln(10\mathrm{e}^{(x-1)/2}) - 2\ln 10 + 1 \\ &= 2(\ln 10 + \ln \mathrm{e}^{(x-1)/2}) - 2\ln 10 + 1 \\ &= 2\ln 10 + 2\left(\frac{x-1}{2}\right) - 2\ln 10 + 1 \\ &= 2\ln 10 + (x-1) - 2\ln 10 + 1 \\ &= x, \end{aligned}$$

so $g(f(x)) = x$ for all x. Therefore, since $f(g(x)) = g(f(x)) = x$ for all x, we see that $g(x) = f^{-1}(x)$.

4. Let $f(t)$ represent the amount of a radioactive substance, in grams, that remains after t hours have passed. Explain the difference between the quantities $f(8)$ and $f^{-1}(8)$ in the context of this problem.

$f(8)$ is the amount, in grams, of the radioactive substance that remains after 8 hours.

On the other hand, $f^{-1}(8)$ is the time that it takes, in hours, until only 8 grams of the radioactive substance remains.